Green's Theorem in the Plane

Let \( C \) be a simple closed curve in \( \mathbb{R}^2 \) which enclosed a domain \( D \). \( P \), \( Q \), \( P_x \), and \( P_y \) are continuous functions on the region \( D \), then

\[
\oint_C P(x, y) \, dx + Q(x, y) \, dy = \iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dA
\]

where \( C \) is oriented positively (counterclockwise) in the integral.

**Proof:**

We first give the proof for a region \( D \) which is both \( x \)-simple and \( y \)-simple.

Let \( C \) be the positively oriented boundary curve of \( D \).

Then \( C = C_1 \cup (-C_2) \) and \( C = C_3 \cup (-C_4) \).

**Parametrizations:**

\[ C_1: (x, y) = (\gamma, f(\tau)), \quad a \leq \tau \leq b \]
\[ C_2: (x, y) = (\gamma, g(\tau)), \quad a \leq \tau \leq b \]

\[ C_3: (x, y) = (h(y), y), \quad c \leq y \leq d \]
\[ C_4: (x, y) = (h(y), y), \quad c \leq y \leq d \]

Using \( C = C_1 \cup (-C_2) \) we can prove

\[
\oint_C P(x, y) \, dx + Q(x, y) \, dy = \iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dA
\]

Using \( C = C_3 \cup (-C_4) \) we can prove

\[
\iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dA
\]

which together give

\[
\oint_C P(x, y) \, dx + Q(x, y) \, dy = \iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dA
\]
Proof of Green's Theorem (continued)

Let's see why \( \oint_C P \, dx = \iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dA \)

\[ D = \left\{ (x, y) : f(\tau) \leq y \leq g(\tau), \ a \leq \tau \leq b \right\} \]

\[ \iint_D \frac{\partial Q}{\partial x} \, dA - \iint_D \frac{\partial P}{\partial y} \, dA = \int_a^b \int_{f(\tau)}^{g(\tau)} \left( Q_x(x, y) - P_y(x, y) \right) \, dy \, d\tau \]

\[ = \int_a^b \left( P(x, g(\tau)) - P(x, f(\tau)) \right) \, d\tau \quad (**) \]

\[ \oint_C P(x, y) \, dx = \int_{C_1} P(x, y) \, dx + \int_{C_2} P(x, y) \, dx \]

Using parametrizations of \( C_1 \) and \( C_2 \)

\[ = \int_a^b P(x, f(\tau)) \, d\tau - \int_a^b P(x, g(\tau)) \, d\tau \]

And \( C_2 \)

\[ = \int_a^b P(x, f(\tau)) - P(x, g(\tau)) \, d\tau \quad (***) \]

Expressions in (**) and (***) are the same.

Therefore \( \oint_C P \, dx = \iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dA \). Similarly.

For a more general region \( D \), we can draw extra curves inside \( D \) to separate \( D \) into a union of regions which are both \( x \)-simple and \( y \)-simple. Green's Thm holds in those smaller regions.

The line integral of \( P \, dx + Q \, dy \) on counterclockwise boundary cancels the extra curves we have drawn, and equals \( \oint_C P \, dx + Q \, dy \) at the end.

Sum of \( \iint_{D_i} \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \, dA \) on all smaller regions \( D_i \) equals \( \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \, dA \).

This proves Green's Thm on \( D \).
Example

Calculate \( \int_C xy \, dx + (x^2 + y^2) \, dy \) where \( C = C_1 \cup C_2 \cup C_3 \) as shown in the figure.

\( C_1, \quad y = e^x \)
\( C_2, \quad y = e \)
\( C_3, \quad 0 \leq x \leq 1 \)

Solution

\( C = C_1 \cup C_2 \cup C_3 \) is a positively oriented simple closed curve which encloses the shaded region \( D \) below.

\( P(x, y) = xy, \quad Q(x, y) = x^2 + y^2 \) and \( \frac{\partial Q}{\partial x} \) and \( \frac{\partial P}{\partial y} \) are continuous on \( \mathbb{R}^2 \), hence on \( D \). (\( \frac{\partial Q}{\partial x} = 2x, \frac{\partial P}{\partial y} = x \))

Then by Green's Theorem, we have

\[
\oint_C pdx + Q \, dy = \iint_D (Q_x - P_y) \, dA = \iint_D 2x - x \, dA = \iint_D x \, dA
\]

\( D: \begin{cases} 0 \leq x \leq 1 \\ e^x \leq y \leq e \end{cases} \)

\[
= \int_0^1 \int_{e^x}^{e} x \, dy \, dx = \int_0^1 \left[ xe^x - e^x \right]_0^1 \, dx
\]

\[
= \left( e - \frac{e^2}{2} - (e - 1) \right) \left|_0^1 \right. = \frac{e}{2} - 1
\]

Therefore

\[
\oint_C xy \, dx + (x^2 + y^2) \, dy = \frac{e}{2} - 1
\]

Note that we calculated the result without parametrizing the curves \( C_1, C_2 \) and \( C_3 \), instead we applied Green's Theorem to convert the line integral over the closed curve \( C \) to a double integral over the enclosed region.
Example
Calculate \( \int_C (x+y^2) \, dx + (\cos(y^2)+x^2) \, dy \)

where \( C \) is the part of the parabola \( y = \frac{x^2}{4} \) from \((-1,0)\) to \((1,0)\).

Solution
\[
\begin{align*}
\text{If we use a parametrization of } C \\
(\tau, y) = (\tau(\tau), y(\tau)) \text{, then the definite} \\
\text{integral corresponding to } \int_C (x+y^2) \, dx + (\cos(y^2)+x^2) \, dy \\
\text{will contain a part } \int \cos(y(\tau)^2) \, y'(\tau) \, d\tau \\
\text{(not possible to find an antiderivative)} \\
\text{integral of elementary functions})
\end{align*}
\]

On the curve \( C_2 \) from \((-1,0)\) to \((1,0)\) along \( y = \frac{x^2}{4} \),
\[
\int_{C_2} P \, dx + Q \, dy \text{ is easier. Here } C_1 \text{ and } C_2 \text{ are from } (-1,0) \text{ to } (1,0) .
\]

\( C_3 = C \cup (-C_2) \) is a simple closed curve which encloses the shaded region \( D \).

\( C \) is counterclockwise oriented and \( P \) and \( Q \) are continuous on \( D \). Then by Green's Theorem,
\[
\begin{align*}
\oint_C P \, dx + Q \, dy &= \iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dA \\
&= \iint_D \left( \frac{\partial}{\partial x} \left( \cos(y^2)+x^2 \right) - \frac{\partial}{\partial y} (x+y^2) \right) \, dA \\
&= \iint_D (2x - 2y) \, dA
\end{align*}
\]

Thus \( \int_C P \, dx + Q \, dy = \int_{C_2} P \, dx + Q \, dy = \int_{C_2} 2x - 2y \, dA \\
C_2 = [y = \frac{x^2}{4}, -1 < \tau < 1, \tau = \tau(\tau), y = \frac{x^2}{4} \text{ and } P(\tau, y) = x+y^2, Q(\tau, y) = \cos(y^2)+x^2] \)

\[
\begin{align*}
\int_0^1 \int_{x^2-1}^{x^2} 2x - 2y \, dy \, dx &= \frac{1}{2} \int_0^1 \left[ x^2 - 2y^2 \right]_{x^2-1}^{x^2} \, dx \\
&= \frac{1}{2} \int_0^1 (x^4 - 2x^2 + 1) \, dx \\
&= \frac{1}{2} \left[ \frac{x^5}{5} - \frac{2x^3}{3} + x \right]_0^1 \\
&= \frac{1}{30}
\end{align*}
\]

Then \( \int_C P \, dx + Q \, dy = \frac{16}{15} \)
Generalization of Green's Theorem

Theorem 1: Let $C$ be a positively oriented simple closed curve in $\mathbb{R}^2$ and let $C_1, C_2, \ldots, C_n$ be positively oriented simple closed curves which all lie in the region enclosed by $C$ such that $C_1, C_2, \ldots, C_n$ are pairwise disjoint and none of them is inside the other.

Let $D$ be the region inside $C$ and outside the curves $C_1, C_2, \ldots, C_n$. If $P(x, y)$ and $Q(x, y)$ are continuous on $D$ such that $Q_y$ and $P_x$ are continuous on $D$, then

$$\sum_{i=1}^{n} \oint_{C_i} P \, dx + Q \, dy = \iint_{D} Q_y \, dx - P_x \, dy = \iint_{D} Q_y - P_x \, dx \, dy$$

Outline of the proof: We'll prove it for $n = 1$. For $n > 1$ similar arguments apply.

Green's Thm on $D_1$:
$$\oint_{C_1} P \, dx + Q \, dy = \iint_{D_1} Q_y - P_x \, dx \, dy$$

Green's Thm on $D_2$:
$$\oint_{C_2} P \, dx + Q \, dy = \iint_{D_2} Q_y - P_x \, dx \, dy$$

Adding up we get:
$$\oint_{C} P \, dx + Q \, dy = \iint_{D_1} Q_y - P_x \, dx \, dy + \iint_{D_2} Q_y - P_x \, dx \, dy$$

$C = C_1 \cup C_3 \cup \ldots \cup C_n$

$C_2 = C_5 \cup C_7$

$C_3 = C_4 \cup C_6 \cup C_7$

$C_4 = C_5 \cup C_7$

$C_5 = C_7$

$C_6 = C_7$
Example

Let \( \vec{F}(x,y) = \left( \frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \right) \)

Calculate \( \oint_{C_1} \vec{F} \cdot d\vec{r} \) and \( \oint_{C_2} \vec{F} \cdot d\vec{r} \) where \( C_1 \) and \( C_2 \) are as shown.

Solution:

\[
Q_x = P_y = -\frac{x^2 + y^2}{(x^2 + y^2)^2} \quad \text{on } \mathbb{R}^2 \setminus \{(0,0)\}
\]

If \( D_2 \) is the region enclosed by \( C_2 \), then \( Q_x = P_y \) on \( D_2 \).

\( P, Q, Q_x \) and \( P_y \) are continuous on \( \mathbb{R}^2 \setminus \{(0,0)\} \), hence on \( D_2 \), too.

Then by Green's Thm:

\[
\oint_{C_2} P \, dx + Q \, dy = \iint_{D_2} (Q_x - P_y) \, dA = 0
\]

\( \oint_{C_2} \vec{F} \cdot d\vec{r} = 0 \) for any simple closed curve \( C_3 \) enclosing a region \( D_3 \) which does not contain \((0,0)\).

Note: We have seen in a previous example that this vector field \( \vec{F} \) is not conservative on \( \mathbb{R}^2 \setminus \{(0,0)\} \). By showing \( \oint_{C_2} \vec{F} \cdot d\vec{r} = 2\pi \neq 0 \) for the unit circle \( C_1: x^2 + y^2 = 1 \), above discussion for \( C_2 \) shows that \( \oint_{C_3} \vec{F} \cdot d\vec{r} = 0 \) for any simple closed curve \( C_3 \) enclosing a region \( D_3 \) which does not contain \((0,0)\).

That is, \((0,0)\) is not inside \( C_3 \) \( \Rightarrow \oint_{C_3} \vec{F} \cdot d\vec{r} = 0 \).

But \((0,0)\) is inside \( C_1 \). \( C_1 \) encloses \( D_2 \) \( \Rightarrow \) \( (0,0) \in D_2 \).

Here \( P, Q, Q_x \) and \( P_y \) are not continuous on \( D_1 \), they are not even defined at \((0,0)\) \( \in D_2 \). Thus, we can't directly apply Green's Thm.

To \( C_1 \) and \( D_2 \): Let \( C_4 \) be circle of radius \( r \) with center \((0,0)\).

Choose \( r \) so small that \( C_4 \) is inside \( C \).

Using parametrization of \( C_4 \):

\[
C_4 : (x,y) = (r \cos t, r \sin t), 0 \leq t \leq 2\pi
\]

\[
\oint_{C_4} P \, dx + Q \, dy = \iint_{D_4} (Q_x - P_y) \, dA = 0
\]

\( \oint_{C_1} \vec{F} \cdot d\vec{r} = 0 \) for any simple closed curve \( C_3 \) enclosing a region \( D_3 \) which does not contain \((0,0)\).

Let \( D_4 \) be the region inside \( C_1 \) and outside \( C_4 \).

Then by generalization of Green's Theorem: (since \( P, Q, Q_x, P_y \) are continuous on \( D_4 \))

\[
\oint_{C_1} \vec{F} \cdot d\vec{r} - \oint_{C_4} \vec{F} \cdot d\vec{r} = \iint_{D_4} (Q_x - P_y) \, dA = 0
\]

\( \oint_{C_1} \vec{F} \cdot d\vec{r} = 0 \) for any simple closed curve \( C_3 \) enclosing a region \( D_3 \) which does not contain \((0,0)\).
An application of Green's Theorem

Let $C$ be a simple closed curve in $\mathbb{R}^2$ oriented counter-clockwise.
Let $D$ be the region enclosed by $C$.

Area of $D = \oint_C y \, dx - x \, dy$.

If $P(x,y)$ and $Q(x,y)$ are continuous on $D$ and if $Q_x$ and $P_y$ are also continuous on $D$, then by Green's Theorem,

$$\oint_C P \, dx + Q \, dy = \iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dA.$$  

If we choose $P(x,y)$ and $Q(x,y)$ such that $P, Q, Q_x$ and $P_y$ are continuous on all of $\mathbb{R}^2$ and $Q_x(x,y) - P_y(x,y) = 1$ for all $(x,y) \in \mathbb{R}^2$, then for any such curve $C$ we get

$$\oint_C P \, dx + Q \, dy = \iint_D Q_x - P_y \, dA = \iint_D 1 \, dA = \text{Area of } D.$$

Table:

<table>
<thead>
<tr>
<th>$P$</th>
<th>$Q$</th>
<th>$Q_x - P_y$</th>
</tr>
</thead>
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<tr>
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<td>1</td>
</tr>
<tr>
<td>$y$</td>
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<td>1</td>
</tr>
<tr>
<td>$-y$</td>
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<td>1</td>
</tr>
<tr>
<td>$-y/2$</td>
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<td>1</td>
</tr>
</tbody>
</table>

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More examples can be found.

**Theorem:** If $C$ is a simple closed curve in $\mathbb{R}^2$ which is oriented counter-clockwise and if $D$ is the region enclosed by $C$, then

$$\text{Area of } D = \text{A}(D) = \oint_C y \, dx - x \, dy = \iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dA = \oint_C \frac{y}{2} \, dx + \frac{x}{2} \, dy.$$
A note for further study at the end of the course.

Due to lack of time, some subjects in vector calculus are not included in the syllabus of Math 120 course. We didn’t cover surface integrals and surface integrals of vector fields.

Many concepts in engineering and physics such as flux involve surface integrals.

As Green’s Theorem gives a relation between a line integral on a curve and a double integral on the enclosed region, the two other big theorems of vector calculus: Divergence Thm and Stokes’ Theorem gives relations between a surface integral over a closed surface and a triple integral on the region enclosed by the surface, and between a surface integral over a surface with boundary and a line integral along the boundary curve.

I suggest you to study these subjects from the textbook.