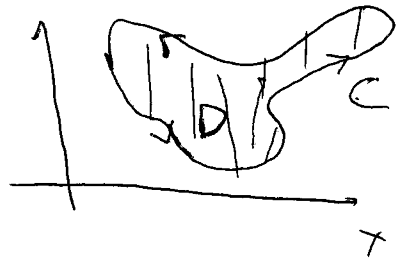
Green's Theorem

Let C be a simple closed curve in $\mathbb{R}^2$ which enclosed a domain D. If P, Q, Q_x and P_y are continuous functions on the region D, then

$$\oint_C P(x,y)\,dx + Q(x,y)\,dy = \iint_D Q_x(x,y) - P_y(x,y)\,dA$$

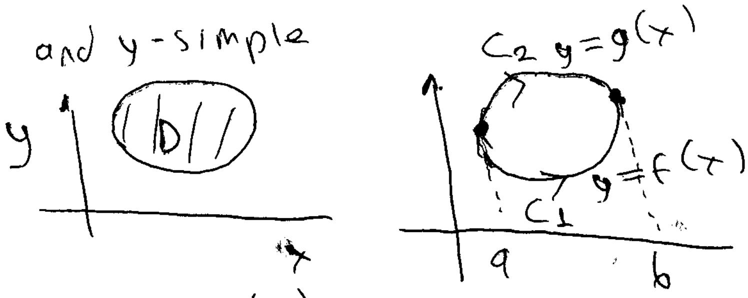
where C is oriented positively (counterclockwise) in the integral

$\oint_C P\,dx + Q\,dy$.

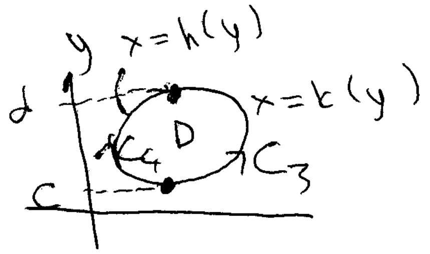


Proof:

We first give the proof for a region D which is both x simple and y-simple



$$D = \begin{cases} a \le x \le b \\ f(x) \le y \le g(x) \end{cases} \quad (D \text{ is } x\text{-simple})$$



$$D = \begin{cases} c \le y \le d \\ h(y) \le x \le k(y) \end{cases} - (D \text{ is } y\text{-simple})$$

Let C be the positively oriented boundary curve of D.

Then $C = C_1 \cup (-C_2)$ and $C = C_3 \cup (-C_4)$

Parametrizations:

$C_1: (x,y) = (x, f(x)),\ a \le x \le b$

$C_2: (x,y) = (x, g(x)),\ a \le x \le b$

Using $C = C_1 \cup (-C_2)$ we can prove

$$\int_C P(x,y)\,dx = \int_{C_1 \cup (-C_2)} P(x,y)\,dx = \iint_D -P_y(x,y)\,dA$$

$C_3: (x,y) = (k(y), y),\ c \le y \le d$

$C_4: (x,y) = (h(y), y),\ c \le y \le d$

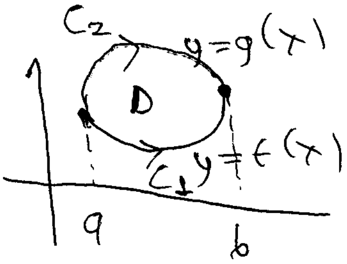
Using $C = C_3 \cup (-C_4)$ we can prove

$$\int_C Q\,dy = \int_{C_3 \cup (-C_4)} Q\,dy = \iint_D Q_x\,dA$$

which together give $\oint_C P\,dx + Q\,dy = \iint_D Q_x - P_y\,dA$.

Proof of Green's Theorem (continued)

Let's see why $\oint_C P\,dx = \iint_D -P_y\,dA$



$$D = \begin{cases} a \le x \le b \\ f(x) \le y \le g(x) \end{cases}$$

$$\iint_D -P_y\,dA = \int_a^b \int_{f(x)}^{g(x)} -P_y(x,y)\,dy\,dx$$

$$= \int_a^b -P(x,g(x)) + P(x,f(x))\,dx \quad (**)$$

$$\oint_C P(x,y)\,dx = \int_{C_1 \cup (-C_2)} P(x,y)\,dx = \int_{C_1} P(x,y)\,dx - \int_{C_2} P(x,y)\,dx$$

using parametrizations of C_1 and C_2

$$= \int_a^b P(x,f(x))\,dx - \int_a^b P(x,g(x))\,dx$$

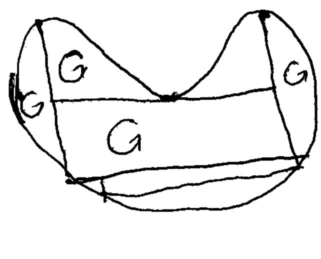
$$= \int_a^b P(x,f(x)) - P(x,g(x))\,dx \quad (***)$$

Expressions in $(**)$ and $(***)$ are the same.

Therefore $\oint_C P\,dx = \iint_D -P_y\,dA$. $\oint_C Q\,dy = \iint_D Q_x\,dA$ is proved similarly.

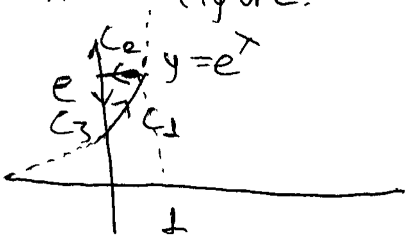
For a more general region D, we can draw extra curves inside D to separate D into a union of regions which are both x-simple and y simple. Green's Thm holds in those smaller regions. Line integral of $P\,dx + Q\,dy$ on counterclockwise boundaries cancel on the extra curves we have drawn, and equals $\oint_C P\,dx + Q\,dy$ at the end. Sum of $\iint_{D_i} Q_x - P_y\,dA$ on all smaller regions D_i equals $\iint_D Q_x - P_y\,dA$.

This proves Green's Thm on D



Example

Calculate $\oint_C xy\,dx+(x^2+y^2)\,dy$ where $C=C_1\cup C_2\cup C_3$ as shown in the figure.

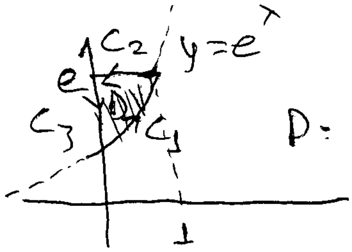


Solution

$C=C_1\cup C_2\cup C_3$ is a positively oriented simple closed curve which encloses the shaded region D below. $P(x,y)=xy$, $Q(x,y)=x^2+y^2$ and Q_x ~~P~~ and P_y are continuous on $\mathbb{R}^2$, hence on D ($Q_x=2x$, $P_y=x$)

Then by Green's Theorem, we have

$$\oint_C P\,dx+Q\,dy=\iint_D Q_x-P_y\,dA=\iint_D 2x-x\,dA=\iint_D x\,dA$$



$$D:\begin{cases}0\le x\le 1\\ e^x\le y\le e\end{cases}\qquad \iint_D x\,dA=\int_0^1\int_{e^x}^{e} x\,dy\,dx$$

$$=\int_0^1 ex-xe^x\,dx$$

$$=\left(e\frac{x^2}{2}-(xe^x-e^x)\right)\Big|_0^1=\frac{e}{2}-1$$

Therefore $\oint_C xy\,dx+(x^2+y^2)\,dy=\frac{e}{2}-1$

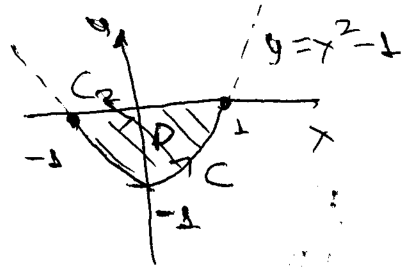
Note that we calculated the result without parametrizing the curves C_1, C_2 and C_3, instead we applied Green's Theorem to convert the line integral over the closed curve C to a double integral on the enclosed region.

Example

Calculate $\int_C (x+y^2)\,dx + (\cos(y^2)+x^2)\,dy$

where C is the part of the parabola $y=x^2-1$ from $(-1,0)$ to $(1,0)$.

Solution



If we use a parametrization of C $(x,y)=(x(t),y(t))$, then the definite integral corresponding to $\int_C (x+y^2)dx+(\cos y^2+x^2)dy$ will contain a part $\int \cos((y(t))^2)\,y'(t)\,dt$

$= \int \cos(u^2)\,du$ where $u=y(t)$ ← (not possible to find an antiderivative in terms of elementary functions)

On the curve C_2 from $(-1,0)$ to $(1,0)$ along x-axis, $\int_{C_2} P\,dx+Q\,dy$ is easier. Here C and C_2 are from $(-1,0)$ to $(1,0)$.

$C_3 = C \cup (-C_2)$ is a simple closed curve which encloses the shaded region D. C_3 is counterclockwise oriented and Q_x and P_y and P and Q are continuous on D. Then by Green's Theorem

$P(x,y)=x+y^2$

$Q(x,y)=\cos y^2+x^2$

$$\oint_{C_3} P\,dx+Q\,dy = \iint_D Q_x - P_y\,dA$$

$$\int_C P\,dx+Q\,dy - \int_{C_2} P\,dx+Q\,dy = \iint_D 2x-2y\,dA$$

Thus $\int_C P\,dx+Q\,dy = \int_{C_2} P\,dx+Q\,dy + \iint_D 2x-2y\,dA$

$C_2: (x,y)=(x,0),\ -1\le x\le 1$

$x=x \quad dx=dx$

$y=0 \quad dy=0\,dx$

$$\int_{C_2} (x+y^2)dx+(\cos y^2+x^2)dy = \int_{-1}^{1} (x+0^2)1+(\cos 0^2+x^2)0\,dx$$

$$=\int_{-1}^{1} x\,dx = 0$$

$D = \begin{cases} -1\le x\le 1 \\ x^2-1\le y\le 0\end{cases} \Rightarrow \iint_D 2x-2y\,dA = \int_{-1}^{1}\int_{x^2-1}^{0} 2x-2y\,dy\,dx$

$$=\int_{-1}^{1} (2xy-y^2)\Big|_{y=x^2-1}^{y=0}dx = \int_{-1}^{1} (x^2-1)^2-2x^3+2x\,dx = \dots = \frac{16}{15}$$

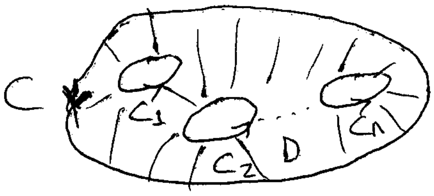
Then $\int_C P\,dx+Q\,dy = 0+\frac{16}{15} = \frac{16}{15}$

A Generalization of Green's Theorem

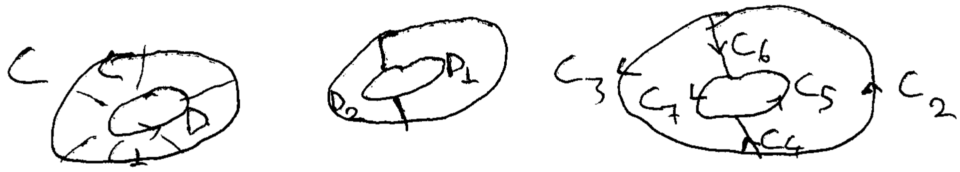
Theorem: Let C be a positively oriented simple closed curve in $\mathbb{R}^2$ and let $C_1, C_2, \dots, C_n$ be positively oriented simple closed curves which all lie in the region enclosed by C such that $C_1, C_2, \dots, C_n$ are pairwise disjoint and none of them is inside the other.
Let D be the region inside C and outside the curves $C_1, C_2, \dots, C_n$
If $P(x,y)$ and $Q(x,y)$ are continous on D such that Q_x and P_y are continous on D, then

$$\oint_C P\,dx + Q\,dy - \sum_{i=1}^{n} \oint_{C_i} P\,dx + Q\,dy = \iint_D Q_x - P_y \, dA$$

$$\oint_C P\,dx + Q\,dy - \oint_{C_1} P\,dx + Q\,dy - \dots - \oint_{C_n} P\,dx + Q\,dy = \iint_D Q_x - P_y \, dA$$



Outline of the proof: We'll prove it for $n=1$. For $n>1$ similar argument applies



Green's Thm on D_1: $\displaystyle\int_{C_2 \cup C_6 \cup (-C_5) \cup (-C_4)} P\,dx + Q\,dy = \iint_{D_1} Q_x - P_y \, dA$

Green's Thm on D_2: $\displaystyle\int_{C_3 \cup C_4 \cup (-C_7) \cup (-C_6)} P\,dx + Q\,dy = \iint_{D_2} Q_x - P_y \, dA$

Adding up we get (integrals over $\pm C_4$ and $\pm C_6$ cancel) = $\displaystyle\int_{(C_2 \cup C_3) \cup (-(C_5 \cup C_7))} P\,dx + Q\,dy = \iint_{D_1} Q_x - P_y \, dA + \iint_{D_2} Q_x - P_y \, dA$

$C = C_2 \cup C_3$
$C_1 = C_5 \cup C_7$

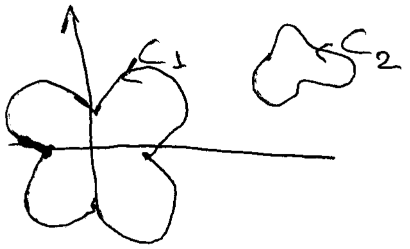
$$\oint_C P\,dx + Q\,dy - \oint_{C_1} P\,dx + Q\,dy = \iint_D Q_x - P_y \, dA$$

$P\,dx + Q\,dy$ $P\,dx + Q\,dy$

Example

Let $\vec{F}(x,y) = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right)$

Calculate $\oint_{C_1} \vec{F}\cdot d\vec{r}$ and $\oint_{C_2} \vec{F}\cdot d\vec{r}$ where C_1 and C_2 are as shown.



$\vec{F} = (P,Q)$ where

$P(x,y) = \frac{-y}{x^2+y^2}$

$Q(x,y) = \frac{x}{x^2+y^2}$

Solution:

$Q_x = P_y = \frac{-x^2+y^2}{(x^2+y^2)^2}$ on $\mathbb{R}^2 - \{(0,0)\}$

If D_2 is the region enclosed by C_2, then $Q_x = P_y$ on D_2.
P, Q, Q_x and P_y are cont. on $\mathbb{R}^2 - \{(0,0)\}$, hence on D_2, too.
Then by Green's Thm:

$$\oint_{C_2} P\,dx + Q\,dy = \iint_{D_2} Q_x - P_y\, dA = \iint_{D_2} 0\, dA = 0$$

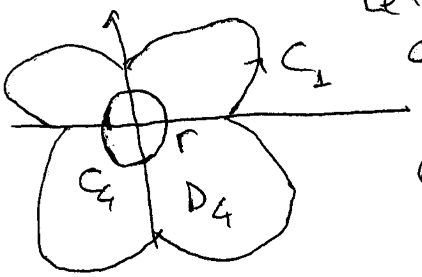
Note: We have seen in a previous example that this vector field $\vec{F}$ is not conservative on $\mathbb{R}^2 - \{(0,0)\}$ by showing $\oint_C \vec{F}\cdot d\vec{r} = 2\pi \neq 0$ for the unit circle $C: x^2+y^2=1$.

Above discussion for C_2 shows that $\int_{C_3} \vec{F}\cdot d\vec{r} = 0$ for any simple closed curve C_3 enclosing a region D_3 which does not contain $(0,0)$

that is, $(0,0)$ is not inside $C_3 \Rightarrow \int_{C_3} \vec{F}\cdot d\vec{r} = 0$.

But $(0,0)$ is inside C_1. C_1 encloses $D_1 \Rightarrow (0,0) \in D_1$.

Here P, Q, Q_x and P_y are not continuous on D_1, they are not even defined at $(0,0) \in D_1$. Thus, we can't directly apply Green's Thm to C_1 and D_1. Let C_4 be circle of radius r with center $(0,0)$.



Let C_4 be circle of radius r with center $(0,0)$. Choose r so small that C_4 is inside C_1.
Using parametrization of C_4:
$C_4: (x,y) = (r\cos t, r\sin t), 0 \le t \le 2\pi$, show that $\oint_{C_4} P\,dx + Q\,dy = 2\pi$

Let D_4 be the region inside C_1 and outside C_4.
Then by generalization of Green's Theorem: (since P, Q, Q_x, P_y are cont on D_4)

$$\oint_{C_1} P\,dx + Q\,dy - \oint_{C_4} P\,dx + Q\,dy = \iint_{D_4} Q_x - P_y\, dA = \iint_{D_4} 0\, dA = 0 \Rightarrow \oint_{C_1} P\,dx + Q\,dy = \oint_{C_4} P\,dx + Q\,dy = 2\pi$$

An application of Green's Theorem,

Let C be a simple closed curve in $\mathbb{R}^2$ oriented counterclockwise.

Let D be the region enclosed by C.

$$\text{Area of } D = \iint_D 1 \, dA.$$

If $P(x,y)$ and $Q(x,y)$ are cont on D and if Q_x and P_y are also continuous on D, then by Green's Thm,

$$\oint_C P\,dx + Q\,dy = \iint_D Q_x - P_y \, dA$$

If we choose $P(x,y)$ and $Q(x,y)$ such that P, Q, Q_x and P_y are continuous on all of $\mathbb{R}^2$ and $Q_x(x,y) - P_y(x,y) = 1$ for all $(x,y) \in \mathbb{R}^2$, then for any such curve C we get

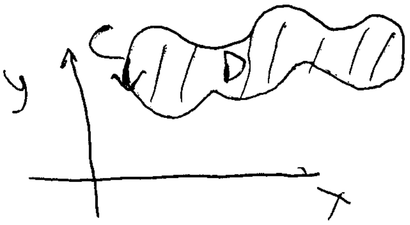
$$\oint_C P\,dx + Q\,dy = \iint_D Q_x - P_y \, dA = \iint_D 1 \, dA = \text{Area of } D$$

P	Q	$Q_x - P_y$
0	x	1
$-y$	0	1
$-\frac{y}{2}$	$\frac{x}{2}$	1

$\rightarrow$ More examples can be found.

Theorem, If C is a simple closed curve in $\mathbb{R}^2$ which is oriented counterclockwise and if D is the region enclosed by C, then

$$\text{Area of } D = A(D) = \oint_C x\,dy = \oint_C -y\,dx = \oint_C -\frac{y}{2}\,dx + \frac{x}{2}\,dy$$



A note for further study at the end of the course

Due to lack of time, some subjects in vector calculus are not included in the syllabus of Math 120 course.
We didn't cover surface integrals and surface integrals of vector fields.
Many concepts in engineering and physics such as flux involve surface integrals.
As Green's Theorem gives a relation between a line integral on a curve and a double integral on the enclosed region, the two other big theorems of vector calculus: Divergence Thm and Stokes' Theorem gives relations between a surface integral over a closed surface and a triple integral on the region enclosed by the surface, and between a surface integral over a surface with boundary and a line integral along the boundary curve.
I suggest you to study these subjects from the textbook.