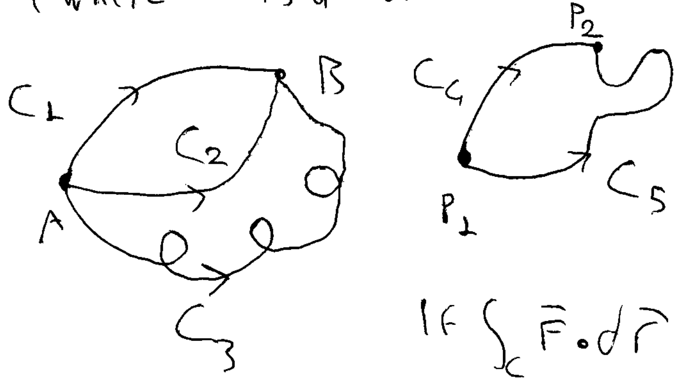
Path Independent Line Integrals

For a vector field $\vec{F}$, the line integral $\int_C \vec{F} \cdot d\vec{r}$ is called path independent in the domain D if for each pair of points A and B in D, the result of $\int_C \vec{F} \cdot d\vec{r}$ is the same for all C in D from point A to point B. In other words the result of $\int_C \vec{F} \cdot d\vec{r}$ depends only the initial point A of C and terminal point B of C, but does not depend on the path C from A to B (where C is a curve which completely lies in D)



If $\int_C \vec{F} \cdot d\vec{r}$ is independent of path in D, then

$$\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_2} \vec{F} \cdot d\vec{r} = \int_{C_3} \vec{F} \, d\vec{r} \text{ and } \int_{C_4} \vec{F} \cdot d\vec{r} = \int_{C_5} \vec{F} \cdot d\vec{r}$$

But $\int_{C_1} \vec{F} \cdot d\vec{r}$ and $\int_{C_4} \vec{F} \cdot d\vec{r}$ may be different

since initial points of C_1 and C_4 are different (also terminal points are different).

• Do we have path independent line integrals?
From physics, gravitational field is path independent.
Question: What characterizes path independent line integrals?

Fundamental Theorem of Line Integrals

If f has continuous first order partial derivatives on D, then

$$\int_C \vec{\nabla f} \cdot d\vec{r} = f(B) - f(A)$$ for any curve C in D where

A: initial point of C and B: terminal point of C.

Proof: (Proof in $\mathbb{R}^3$ is given, proof for $\mathbb{R}^2$ is similar)

Let C be a curve in D parametrized by

$$C: (x,y,z) = \vec{r}(t) = (x(t), y(t), z(t)), \; a \leq t \leq b$$

$$\int_C \vec{\nabla f} \cdot dr = \int_C f_x\,dx + f_y\,dy + f_z\,dz$$

$$= \int_a^b f_x(x(t),y(t),z(t))x'(t) + f_y(x(t),y(t),z(t))y'(t) + f_z(x(t),y(t),z(t))z'(t)\,dt$$

$$= \int_a^b \frac{d}{dt} f(x(t),y(t),z(t))\,dt \quad \text{(By Chain Rule)}$$

$$= f(x(t),y(t),z(t))\Big|_a^b \quad \text{(by F.T.C.)}$$

$$= f(x(b),y(b),z(b)) - f(x(a),y(a),z(a))$$

$$= f(B) - f(A)$$ where $B = \vec{r}(b)$: terminal point of C

$A = \vec{r}(a)$: initial point of C.

Theorem:

$\int_C \vec{F}\cdot d\vec{r}$ is path independent on a domain D if and only if $\vec{F}$ is a conservative vector field in D.

When ϕ is a potential function of the conservative vector field $\vec{F}$ in D (when $\vec{F} = \vec{\nabla}\phi$ on D), we have

$$\int_C \vec{F}\cdot d\vec{r} = \phi(B) - \phi(A)$$ where A: initial point of C, B: terminal point of C.

Proof:

($\Leftarrow$): Assume $\vec{F}$ is conservative on D such that $\vec{F} = \vec{\nabla}\phi$ on D.

Then for any curve C in D, using the previous theorem we get

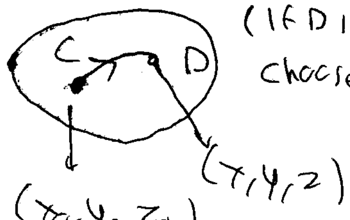
$$\int_C \vec{F}\cdot d\vec{r} = \int_C \vec{\nabla}\phi \cdot d\vec{r} = \phi(B) - \phi(A).$$

Hence, result of $\int_C \vec{F}\cdot d\vec{r}$ depends only on initial and terminal points A and B of C. Thus $\int_C \vec{F}\cdot d\vec{r}$ is path independent in D.

($\Rightarrow$) Assume $\int_C \vec{F}\cdot d\vec{r}$ is path independent in D.

If D is connected (any two points in D can be joined by a curve in D) by choosing a fixed point $(x_0, y_0, z_0) \in D$, we can define a function $\phi(x,y,z) = \int_C \vec{F}\cdot d\vec{r}$ where C is any curve in D from (x_0,y_0,z_0) to (x,y,z)

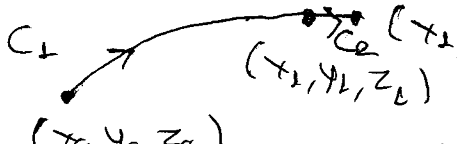
(If D is not connected, choose a pt in components of D)



(No matter which such C is chosen, result of $\int_C \vec{F}\cdot d\vec{r}$ is the same by path independence)

Then, we can show that $\vec{\nabla}\phi = \vec{F}$ on D.

Let $\vec{F} = (P, Q, R)$, to see $\phi_y = Q$:



$$\phi_y(x_1,y_1,z_1) = \lim_{h\to 0} \frac{\phi(x_1,y_1+h,z_1) - \phi(x_1,y_1,z_1)}{h}$$

$$= \lim_{h\to 0} \frac{\int_{C_1\cup C_2} \vec{F}\cdot d\vec{r} - \int_{C_1}\vec{F}\cdot d\vec{r}}{h} = \lim_{h\to 0}\frac{\int_{C_2}\vec{F}\cdot d\vec{r}}{h} = \lim_{h\to 0}\frac{\int_0^h Q(x_1,y_1+t,z_1)\,dt}{h} = Q(x_1,y_1,z_1)$$

We used param of C_2, FTC at the end $\rightarrow$

Theorem

The line integral $\int_C \vec{F} \cdot d\vec{r}$ is independent of path on the domain D

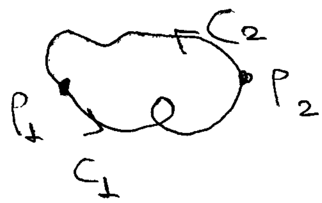
if and only if $\oint_C \vec{F} \cdot d\vec{r} = 0$ for all closed curves C which lie in D.

Proof:

$(\Rightarrow)$: Assume $\int_C \vec{F} \cdot d\vec{r}$ is path independent on D.

Let C be a closed curve in D. Choosing points P_1 and P_2 on C,

we can write $C = C_1 \cup C_2$



C_1: from P_1 to P_2, C_2: from P_2 to P_1

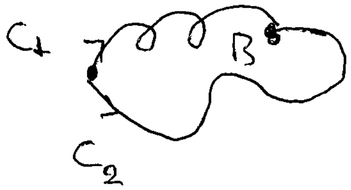
$-C_2$: from P_1 to P_2

Using path indepodence on D, we get $\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{-C_2} \vec{F} \cdot d\vec{r} = -\int_{C_2} \vec{F} \cdot d\vec{r}$

Thus $\oint_C \vec{F} \cdot d\vec{r} = \int_{C_1 \cup C_2} \vec{F} \cdot d\vec{r} = \int_{C_1} \vec{F} \cdot d\vec{r} + \int_{C_2} \vec{F} \cdot d\vec{r} = -\int_{C_2} \vec{F} \cdot d\vec{r} + \int_{C_2} \vec{F} \cdot d\vec{r} = 0$

$(\Leftarrow)$: Assume now $\oint_C \vec{F} \cdot d\vec{r} = 0$ for all closed curves C which lie in D.

If C_1 and C_2 are curves from point A to point B in D,



then $C = C_1 \cup -C_2$ is a closed curve in D, hence

$\oint_C \vec{F} \cdot d\vec{r} = 0$ due to our assumption.

$$\int_{C_1 \cup -C_2} \vec{F} \cdot d\vec{r} = 0$$

$$\int_{C_1} \vec{F} \cdot d\vec{r} + \int_{-C_2} \vec{F} \cdot d\vec{r} = 0$$

$$\int_{C_1} \vec{F} \cdot d\vec{r} - \int_{C_2} \vec{F} \cdot d\vec{r} = 0 \Rightarrow \int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_2} \vec{F} \cdot d\vec{r}$$

Since C_1 and C_2 are arbitrarily chosen curves from A to B in D,
this means that $\int_C \vec{F} \cdot d\vec{r}$ is path independent in D.

Combining the previous two theorems, we can write:

Theorem, The following statements are equivalent:
(If one of them is true, all three statements are true)

1) $\vec{F}$ is a conservative vector field on a domain D.

2) The line integral $\int_C \vec{F} \cdot d\vec{r}$ is path independent in D.

3) $\oint_C \vec{F} \cdot d\vec{r} = 0$ for all closed curves C which lie in D.

Example, Calculate $\int_C (e^{-x^2} + y)\,dx + (x+1)\,dy$ where C is the right half of the ellipse $\frac{x^2}{4} + y^2 = 1$ from $(0,-1)$ to $(0,1)$.

Solution: $\vec{F}(x,y) = (P, Q) = (e^{-x^2} + y, x+1)$ is defined on $D = \mathbb{R}^2$

- $Q_x = 1 = P_y \Rightarrow Q_x = P_y$ on D.
- $D = \mathbb{R}^2$ is simply connected

$\Big\} \Rightarrow \vec{F}$ is a conservative vector field on $D = \mathbb{R}^2$.

Therefore $\int_C \vec{F} \cdot d\vec{r}$ is path independent.

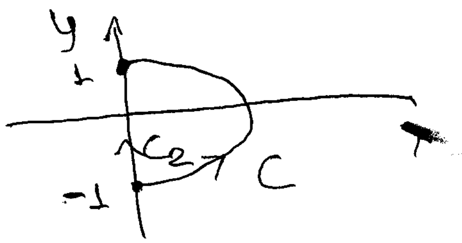
Then, $\int_C \vec{F} \cdot d\vec{r} = \int_{C_2} \vec{F} \cdot d\vec{r}$ for any curve C_2 from $(0,-1)$ to $(0,1)$.

Instead of using the more complicated curve C, let C_2 be the line segment from $(0,-1)$ to $(0,1)$. $C_2 : (x,y) = (0, t), -1 \leq t \leq 1$

$$\int_C (e^{-x^2} + y)\,dx + (x+1)\,dy = \int_{C_2} (e^{-x^2} + y)\,dx + (x+1)\,dy$$

$x = 0 \quad dx = 0\,dt \quad y = t \quad dy = dt$

$$= \int_{-1}^{1} (e^{-0^2} + t)\cdot 0 + (0+1)\cdot 1\,dt = \int_{-1}^{1} 1\,dt = 2$$



Example,

Is the vector field $\vec{F}(x,y) = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\right)$ conservative on $\mathbb{R}^2 - \{(0,0)\}$?

Solution: Let $\vec{F}(x,y) = (P(x,y), Q(x,y))$

$$= \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\right), \ \mathrm{Dom}(\vec{F}) = \mathbb{R}^2 - \{(0,0)\}$$

$$Q_x = \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) = \frac{-x^2+y^2}{(x^2+y^2)^2}, \quad P_y = \frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right) = \frac{-x^2+y^2}{(x^2+y^2)^2}$$

Then, $Q_x = P_y$ on $D = \mathbb{R}^2 - \{(0,0)\}$

Warning: $D = \mathbb{R}^2 - \{(0,0)\}$ is NOT simply connected (D is in $\mathbb{R}^2$ and D has a hole). Even if $Q_x = P_y$ on D, we cannot conclude that $\vec{F}$ is conservative on D since D is not simply connected. The above discussion leads to no conclusion so far.

We know that $\vec{F}$ is conservative on D if and only if $\oint_C \vec{F} \cdot d\vec{r} = 0$ for all closed curves C which lie in D.

If we can find a closed curve C in $D = \mathbb{R}^2 - \{(0,0)\}$ such that $\oint_C \vec{F} \cdot d\vec{r} \neq 0$, then we can conclude that $\vec{F}$ is NOT conservative on D. Let C be the unit circle $x^2+y^2=1$ oriented counterclockwise. A parametrization of C is: (C is a closed curve in D)

$C: (x,y) = (\cos t, \sin t), \ 0 \leq t \leq 2\pi$

Then

$$\oint_C \vec{F} \cdot d\vec{r} = \oint_C \frac{-y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy = \int_0^{2\pi} -\frac{\sin t}{1}(-\sin t) + \frac{\cos t}{1}\cos t\,dt$$

$x = \cos t \quad dx = -\sin t\,dt$

$y = \sin t \quad dy = \cos t\,dt$

$x^2+y^2 = \cos^2 t + \sin^2 t = 1$

C is a closed curve in D

$$= \int_0^{2\pi} \sin^2 t + \cos^2 t\,dt$$

$$= \int_0^{2\pi} 1\,dt = 2\pi \neq 0$$

Conclusion

Since $\oint_C \vec{F} \cdot d\vec{r} \neq 0$, $\vec{F}$ is not conservative on D

Example,

Let $\vec{F}(x,y,z)=(e^y+2xz,\ xe^y+z^2+1,\ x^2+2yz+z^2)$ and C be the curve parametrized by $C:(x,y,z)=\vec{r}(t)=(\cos(\sin t),\tan(t^2),e^{t^3})$, $0\le t\le 1$. If the direction on C is the direction given by the above parametrization, then

a) Express $\int_C \vec{F}\cdot d\vec{r}$ as a definite integral using the given parametrization

b) Calculate $\int_C \vec{F}\cdot d\vec{r}$ by finding a potential function ϕ of $\vec{F}$ if it exists.

Solution, a) $x=\cos(\sin t)$ $\quad dx=-\sin(\sin t)\cdot\cos t\,dt$

$y=\tan(t^2)$ $\quad dy=\sec^2(t^2)\,2t\,dt$

$z=e^{t^3}$ $\quad dz=e^{t^3}\cdot 3t^2\,dt$

$$\int_C \vec{F}\cdot d\vec{r}=\int_C (e^y+2xz)dx+(xe^y+z^2+1)dy+(x^2+2yz+z^2)dz$$

$$=\int_0^1 \Big(e^{\tan(t^2)}+2\cos(\sin t)e^{t^3}\Big)\cdot(-\sin(\sin t)\cdot\cos t)$$
$$+\Big(\cos(\sin t)\,e^{\tan(t^2)}+e^{2t^3}+1\Big)\sec^2(t^2)\cdot 2t$$
$$+\Big(\cos^2(\sin t)+2\tan(t^2)e^{t^3}+e^{2t^3}\Big)e^{t^3}\cdot 3t^2\ dt$$

(a quite lengthy and complicated integral to compute!!)

b) If $\vec{F}$ is a conservative vector field with a potential function $\phi(x,y,z)$, then $\vec{F}=\vec{\nabla}\phi$ and we have:

$$\int_C \vec{F}\cdot d\vec{r}=\int_C \vec{\nabla}\phi\cdot d\vec{r}=\phi(B)-\phi(A) \text{ where}$$

$A=\vec{r}(0)=(\cos(\sin 0),\tan(0^2),e^{0^3})=(1,0,1)$ (initial point of C), and

$B=\vec{r}(1)=(\cos(\sin 1),\tan 1,e)$: terminal point of C.

But not every vector field $\vec{F}$ is conservative. If $\vec{F}$ is not conservative we can't apply this formula. Let's find a potential function of $\vec{F}$, if we

Solution (continued)

Is there $\phi(x,y,z)$ such that $\vec{F} = \vec{\nabla}\phi$?

$\vec{F} = \vec{\nabla}\phi \iff (P,Q,R) = (\phi_x, \phi_y, \phi_z)$

$\iff f_x = P = e^y + 2xz,$

$f_y = Q = xe^y + z^2 + 1,$

$f_z = x^2 + 2yz + z^2$

$\phi_x(x,y,z) = e^y + 2xz \Rightarrow \phi(x,y,z) = \int e^y + 2xz\,dx = xe^y + x^2z + C(y,z)$

Then

$\phi_y(x,y,z) = xe^y + z^2 + 1$

$\frac{\partial}{\partial y}\left(xe^y + x^2z + C(y,z)\right) = xe^y + z^2 + 1$

$xe^y + 0 + C_y(y,z) = xe^y + z^2 + 1 \Rightarrow C_y(y,z) = z^2 + 1$

$\Rightarrow C(y,z) = yz^2 + y + D(z)$

(Note here that if we found that $C_y(y,z)$ depends on x, we would say no such $C(y,z)$ exists, hence no such $\phi(x,y,z)$ exists)

Then, using $\phi_z = x^2 + 2yz + z^2$

$\frac{\partial}{\partial z}\left(xe^y + x^2z + yz^2 + y + D(z)\right) = x^2 + 2yz + z^2$

$0 + x^2 + 2yz + 0 + D'(z) = x^2 + 2yz + z^2$

$\Rightarrow D'(z) = z^2 \Rightarrow D(z) = \int z^2\,dz = \frac{z^3}{3} + E$ (where $E \in \mathbb{R}$ is a constant)

(Since $D(z)$ is only a function of z, $D'(z)$ can depend only on z.
If we had found $D'(z)$ depends on x or y, we would have said no such D exists, and no such ϕ exists)

Conclusion: There is a function $\phi(x,y,z)$ such that $\vec{F} = \vec{\nabla}\phi$, and all such ϕ are given as:

$\phi(x,y,z) = xe^y + x^2z + y^2z + y + \frac{z^3}{3} + E$ where $E \in \mathbb{R}$ is a constant.

Therefore

$\int_C \vec{F} \cdot d\vec{r} = \phi(B) - \phi(A) = \phi(\cos(\sin 1), \tan 1, e) - \phi(1,0,1)$

= substitute the numbers to obtain the result.