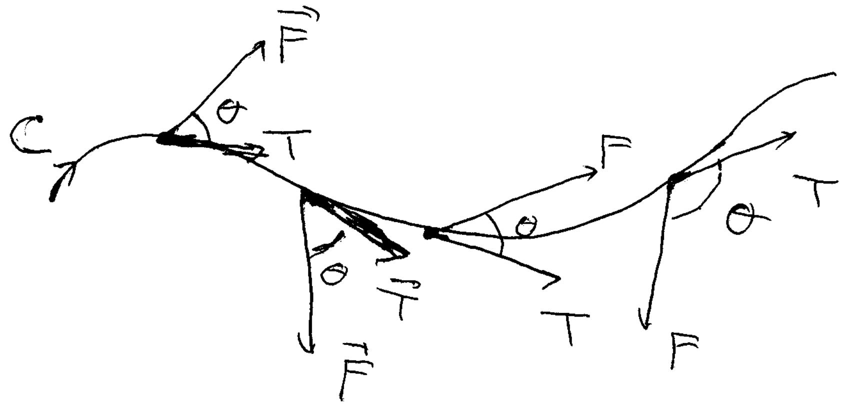
15.4 | Line Integrals of Vector Fields

Let C be a smooth curve in $\mathbb{R}^2$ or $\mathbb{R}^3$ with a given direction on C.
Let $\vec{F}$ be a continuous vector field (component functions are continuous) defined at every point of C.
At every point of C, there is a unique unit tangent vector $\vec{T}$ which shows the given direction on C. $|\vec{T}|=1$.



$\vec{F}$, $\vec{T}$ and the angle θ between $\vec{F}$ and $\vec{T}$ vary on C continuously.

$\vec{F}\cdot\vec{T} = |\vec{F}|\cdot|\vec{T}|\cdot\cos\theta = |\vec{F}|\cdot\cos\theta$ (since $|\vec{T}|=1$)

$\vec{F}\cdot\vec{T}$: Component of the vector field $\vec{F}$ along the tangent line to C in the given direction

: Scalar projection of $\vec{F}$ onto $\vec{T}$.

We define the line integral of a vector field $\vec{F}$ on a curve C with a given direction as:

$$\int_C \vec{F}\cdot d\vec{r} = \underbrace{\int_C \vec{F}\cdot\vec{T}\, ds}_{\text{line integral of the scalar function } \vec{F}\cdot\vec{T} \text{ on } C.}$$

Interpretation:

If $\vec{F}$ is the force vector field which moves an object along a curve C, then $\vec{F} \cdot \vec{T}$ is the component of the force vector in the direction of the motion.

Then, total work done on this object by the force $\vec{F}$ to move it along the curve C in the given direction is:

$$W = \int_C \vec{F} \cdot d\vec{r}$$

* Let $-C$ be the curve C with the opposite direction.

Then

$$\int_{-C} \vec{F} \cdot d\vec{r} = -\int_C \vec{F} \cdot d\vec{r}$$

since if $\vec{T}$ is unit tangent vector along C compatible with the direction of C, then unit tangent vector along $-C$ compatible with the direction of $-C$ is $-\vec{T}$ (opposite direction)

Hence

$$\int_{-C} \vec{F} \cdot d\vec{r} = \int_{-C} \vec{F} \cdot (-\vec{T})\,ds = \int_{-C} -(\vec{F} \cdot \vec{T})\,ds$$

$$= -\int_{-C} \vec{F} \cdot \vec{T}\,ds = -\int_C \vec{F} \cdot \vec{T}\,ds = -\int_C \vec{F} \cdot d\vec{r}$$

C and $-C$ are the same curve with opposite direction, and direction on C does not change result of a line integral of a function (scalar)

Calculating $\int_C \bar{F} \cdot d\bar{r}$ using a smooth parametrization of C

Let $\bar{F}(x,y,z) = (P(x,y,z), Q(x,y,z), R(x,y,z))$

and let C be parametrized by the smooth parametrization

$C: \bar{r}(t) = (x(t), y(t), z(t)),\ a \le t \le b$

such that direction of the parametrization $\bar{r}(t)$ is the same as the given direction on C.

Then $\vec{T} = \dfrac{\bar{r}'(t)}{|\bar{r}'(t)|}$: unit tangent vector along C compatible with given direction.

and

$$\int_C \bar{F} \cdot d\bar{r} = \int_C \bar{F} \cdot \bar{T}\, ds = \int_a^b \left(F \cdot \frac{\bar{r}'(t)}{|\bar{r}'(t)|}\right) \cdot |\bar{r}'(t)|\, dt$$

$ds = |\bar{r}'(t)|\, dt$

$$= \int_a^b \frac{(P, Q, R) \cdot (x'(t), y'(t), z'(t))}{\cancel{|\bar{r}'(t)|}} \cdot \cancel{|\bar{r}'(t)|}\, dt$$

$$= \int_a^b P(x(t), y(t), z(t)) \cdot x'(t) + Q(x(t), y(t), z(t))\, y'(t) + R(x(t), y(t), z(t)) \cdot z'(t)\, dt$$

$$= \int_a^b P \cdot x'(t) + Q \cdot y'(t) + R \cdot z'(t)\, dt$$ where P, Q and R are expressed in terms of t substituting $x = x(t)$, $y = y(t)$ and $z = z(t)$

Notation: For $\vec{F}(x,y,z)=(P(x,y,z),Q(x,y,z),R(x,y,z))$

$$\int_C \vec{F}\cdot d\vec{r}=\int_C P(x,y,z)\,dx+Q(x,y,z)\,dy+R(x,y,z)\,dz$$

$$\boxed{\begin{array}{l}\vec{r}=(x,y,z)\\ d\vec{r}=(dx,dy,dz)\end{array}}=\int_C P\,dx+Q\,dy+R\,dz$$

To calculate $\int_C \vec{F}\cdot d\vec{r}=\int_C P\,dx+Q\,dy+R\,dz$ using the parametrization $C: \vec{r}(t)=(x(t),y(t),z(t)),\ a\leq t\leq b,$

$$\int_C \vec{F}\cdot d\vec{r}=\int_C P\,dx+Q\,dy+R\,dz=\int_a^b P\cdot x'(t)+Q\cdot y'(t)+R\cdot z'(t)\,dt$$

$x=x(t)$ $dx=x'(t)dt$

$y=y(t)$ $dy=y'(t)dt$

$z=z(t)$ $dz=z'(t)dt$

where $P(x,y,z)$, $Q(x,y,z)$ and $R(x,y,z)$ are expressed in terms of t by substituting $x(t),y(t),z(t)$ for x,y and z.

Remark:

$\int_C P\,dx=\int_C \vec{F}\cdot d\vec{r}$ where $\vec{F}=(P,0,0)$

$\int_C Q\,dy=\int_C \vec{F}\cdot d\vec{r}$ where $\vec{F}=(0,Q,0)$

$\int_C R\,dz=\int_C \vec{F}\cdot d\vec{r}$ where $\vec{F}=(0,0,R)$

$\int_C P\,dx+R\,dz=\int_C \vec{F}\cdot d\vec{r}$ where $\vec{F}=(P,0,R)$

Similarly, for C a smooth curve in $\mathbb{R}^2$

parametrized by $C: \bar{r}(t) = (x(t), y(t)), a \le t \le b$,

and $\bar{F}(x,y) = (P(x,y), Q(x,y))$,

$$\int_C \bar{F} \cdot d\bar{r} = \int_C P\,dx + Q\,dy \qquad \begin{array}{l} x = x(t),\ dx = x'(t)dt \\ y = y(t),\ dy = y'(t)dt \end{array}$$

$$= \int_a^b P(x(t),y(t)) \cdot x'(t) + Q(x(t),y(t)) \cdot y'(t)\,dt$$

For a piecewise smooth curve $C = C_1 \cup C_2 \cup \cdots \cup C_n$ with a given direction on C where each C_i is a smooth curve we have

$$\int_C \bar{F} \cdot d\bar{r} = \int_{C_1} \bar{F} \cdot d\bar{r} + \int_{C_2} \bar{F} \cdot d\bar{r} + \cdots + \int_{C_n} \bar{F} \cdot d\bar{r}.$$

Example

$C: \bar{r}(t) = (t^2, \ln(t), \sin(t)), \; 2 \le t \le 7$

Calculate $\int_C \bar{F} \cdot d\bar{r}$ where $\bar{F}(x,y,z) = (x+yz, xy^2z^3, x^2+y^2+z^2)$

Solution: Using the given smooth parametrization of C:

$$\int_C \bar{F} \cdot d\bar{r} = \int_C P\,dx + Q\,dy + R\,dz \quad (\text{where } \bar{F}(x,y,z) = (P,Q,R))$$

$x = t^2 \quad dx = 2t\,dt$

$y = \ln t \quad dy = \frac{1}{t}\,dt$

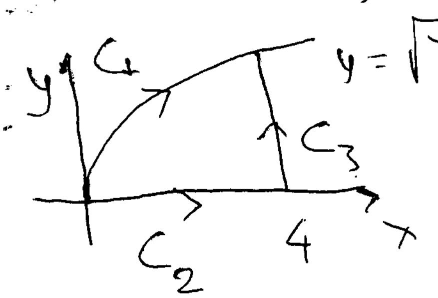
$z = \sin t \quad dz = \cos t\,dt$

$$= \int_C (x+yz)\,dx + xy^2z^3\,dy + (x^2+y^2+z^2)\,dz$$

$$= \int_2^7 (t^2 + \ln t \sin t)\,2t + t^2 (\ln t)^2 \sin^3 t \cdot \frac{1}{t} + (t^4 + (\ln t)^2 + \sin^2 t)\cos t\,dt$$

Ex, Calculate $\int_{C_1} \bar{F} \cdot d\bar{r}$ and $\int_{C_2 \cup C_3} \bar{F} \cdot d\bar{r}$ where $\bar{F}(x,y) = y^2 \cdot \vec{i} + x^2 \cdot \vec{j}$

and C_1, C_2, C_3 are the curves shown in the figure.



Solution:

Parametrizations of the curves with given directions:

$C_1: \bar{r}(t) = (x, \sqrt{x}), \; 0 \le x \le 4$

$C_2: (x,y) = (x, 0), \; 0 \le x \le 4$

$C_3: (x,y) = (4, y), \; 0 \le y \le 2$

$$\int_{C_1} \bar{F} \cdot d\bar{r} = \int_{C_1} y^2\,dx + x^2\,dy = \int_0^4 (\sqrt{x})^2 \cdot 1 + x^2 \cdot \frac{1}{2\sqrt{x}}\,dx = \int_0^4 x + \frac{1}{2} x^{3/2} = \frac{72}{5}$$

$x = x \quad dx = 1\,dx$

$y = \sqrt{x} \quad dy = \frac{1}{2\sqrt{x}}\,dx$

$$\int_{C_2 \cup C_3} \bar{F} \cdot d\bar{r} = \int_{C_2} y^2\,dx + x^2\,dy + \int_{C_3} y^2\,dx + x^2\,dy = \int_0^4 0^2 \cdot 1 + x^2 \cdot 0\,dx$$

$x = x \quad dx = 1\,dx$

$y = 0 \quad dy = 0\,dx$

$x = 4 \quad dx = 0\,dy$

$y = y \quad dy = 1\,dy$

$$+ \int_0^2 y^2 \cdot 0 + 16 \cdot 1\,dy$$

$$= 0 + 32 = 32$$