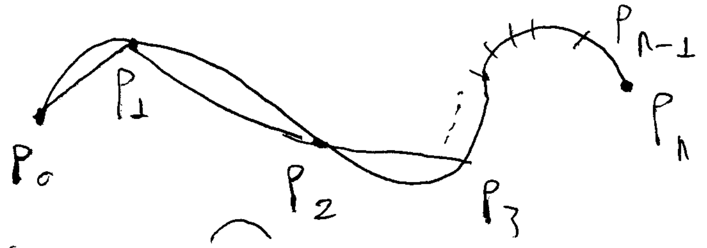
## Arclength

Let $C: \vec{r}(t) = (x(t), y(t), z(t))$, $a \leq t \leq b$ be a curve given by a smooth parametrization.

For $a = t_0 < t_1 < \dots < t_n = b$ a partition of $[a,b]$, let $P_i = \vec{r}(t_i) \in C$, $i = 0, 1, \dots, n$.




If $\Delta s_i = |\overset{\frown}{P_{i-1} P_i}|$ = arclength of the arc on $C$ between $P_{i-1}$ and $P_i$,

$$\Delta s_i \approx |P_{i-1} P_i| = \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2 + (\Delta z_i)^2} = \sqrt{\left(\frac{\Delta x_i}{\Delta t_i}\right)^2 + \left(\frac{\Delta y_i}{\Delta t_i}\right)^2 + \left(\frac{\Delta z_i}{\Delta t_i}\right)^2} \cdot \Delta t_i$$

$L$: arclength of $C \Rightarrow L \approx \sum_{i=1}^{n} |P_{i-1} P_i|$

$$L = \lim_{n \to \infty} \sum_{i=1}^{n} \sqrt{\left(\frac{\Delta x_i}{\Delta t_i}\right)^2 + \left(\frac{\Delta y_i}{\Delta t_i}\right)^2 + \left(\frac{\Delta z_i}{\Delta t_i}\right)^2} \, \Delta t_i$$

(limit is taken as $n \to \infty$ and norm of the partition $\to 0$)

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} \, dt$$

$$L = \int_a^b |\vec{r}\,'(t)| \, dt$$

$$L = \int_{t=a}^{t=b} ds$$

where $\vec{r}\,'(t) = (x'(t), y'(t), z'(t))$

($|\vec{r}\,'(t)|$ = speed of parametriz.)

where $ds = |\vec{r}\,'(t)| \, dt$ is the arclength element for $C$.

Note: Using different parametrizations of $C$ give the same result for arclength.

Example:

Show that the curve $C$ given by the parametrization

$\vec{r}(t) = (t\cos t,\ t\sin t,\ t)$, $t \in \mathbb{R}$ lies on the cone

$z^2 = x^2 + y^2$ and calculate the arclength of the part of $C$

between $(-\pi, 0, \pi)$ and $(4\pi, 0, 4\pi)$.

Solution:

$(x,y,z) \in C \Rightarrow x = t\cos t,\ y = t\sin t,\ z = t$

$$\Rightarrow x^2 + y^2 = t^2\cos^2 t + t^2 \sin^2 t$$

$$= t^2(\cos^2 t + \sin^2 t) = t^2 = z^2$$

$$\Rightarrow x^2 + y^2 = z^2 \Rightarrow C \text{ lies on this cone.}$$

Note that $(-\pi, 0, \pi) = \vec{r}(\pi)$ and $(4\pi, 0, 4\pi) = \vec{r}(4\pi)$, thus

$$L = \int_\pi^{4\pi} |\vec{r}\,'(t)|\, dt = \int_\pi^{4\pi} \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}\, dt$$

$$= \int_\pi^{4\pi} \sqrt{(\cos t - t\sin t)^2 + (\sin t + t\cos t)^2 + 1^2}\, dt$$

$$= \int_\pi^{4\pi} \sqrt{2 + t^2}\, dt = \text{---- apply integration techniques.}$$

$t = \sqrt{2}\tan\theta$

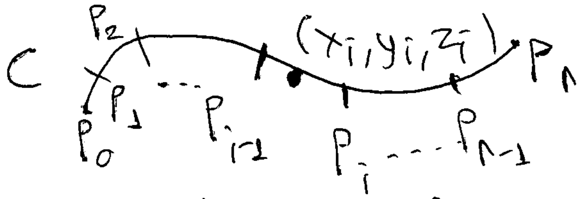
$dt = \sqrt{2}\sec^2\theta\, d\theta$

## 15.3 | Line Integrals of Functions

Let $f(x,y,z)$ be a continuous function defined on every point of a smooth curve $C$.
The line integral $\int_C f(x,y,z)\,ds$ of $f(x,y,z)$ along the curve $C$ is defined as follows:
If $P_0, P_1, \ldots, P_n$ are points on the curve $C$ which divide $C$ into $n$ arcs $\widehat{P_0P_1}, \widehat{P_1P_2}, \ldots, \widehat{P_{n-1}P_n}$, let $\Delta s_i$ = arclength of $\widehat{P_{i-1}P_i}$ and let $(x_i, y_i, z_i)$ be a chosen point on the $i^{th}$ arc $\widehat{P_{i-1}P_i}$.

Then
$$\int_C f(x,y,z)\,ds = \lim_{\substack{n\to\infty \\ \max\{\Delta s_i\}\to 0}} \sum_{i=1}^{n} f(x_i,y_i,z_i)\cdot\Delta s_i$$

where limit is taken as number of subarcs $n$ goes to $\infty$ and maximum of arclengths of the subarcs goes to 0.




Calculation of $\int_C f(x,y,z)\,ds$ :

For a smooth parametrization of $C$
$\bar{r}(t) = (x,y,z) = (x(t), y(t), z(t))$, $a \le t \le b$,

$\Delta s_i \approx |\bar{r}'(t)|\,\Delta t_i$, and the above limit gives the definite integral:
$$\int_C f(x,y,z)\,ds = \int_a^b f(x(t),y(t),z(t))\cdot|\bar{r}'(t)|\,dt$$

$$= \int_a^b f(x(t),y(t),z(t))\sqrt{(x'(t))^2+(y'(t))^2+(z'(t))^2}\,dt$$

Similarly, for a curve $C$ in $\mathbb{R}^2$ parametrized as

$$C: \vec{r}(t) = (x(t), y(t)), \quad a \le t \le b$$

and a continuous function $f(x,y)$ defined on all points of $C$ we have:

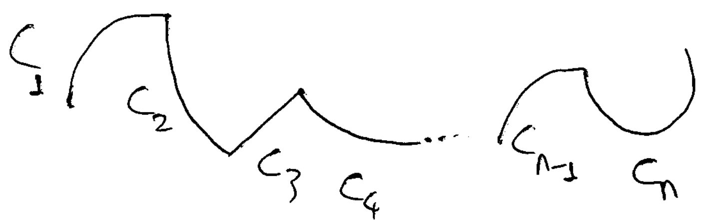
$$\int_C f(x,y)\,ds = \int_a^b f(x(t), y(t)) \cdot |\vec{r}\,'(t)|\,dt$$

$$= \int_a^b f(x(t), y(t)) \cdot \sqrt{(x'(t))^2 + (y'(t))^2}\,dt$$

Note here $ds = |\vec{r}\,'(t)|\,dt$ is the arclength element.

For a piecewise smooth curve $C$ where $C = C_1 \cup C_2 \cup \dots \cup C_n$ such that each $C_i$ is smooth, we have

$$\int_C f(x,y,z)\,ds = \int_{C_1} f(x,y,z)\,ds + \int_{C_2} f(x,y,z)\,ds + \dots \int_{C_n} f(x,y,z)\,ds$$




Notation $\oint_C f(x,y,z)\,ds$ means $C$ is a closed curve.

Remark: Result of the line integral $\int_C f(x,y,z)\,ds$ does not depend on the parametrization chosen to calculate it.
But note that a smooth parametrization of $C$ should be used, and if $C$ is closed, the parametrization should traverse the curve only once (it should not wind around it several times).

Example

$C: (x,y,z) = \bar{r}(t) = (t^3, e^t, \sin t), \quad 0 \le t \le 2$

Then $\displaystyle\int_C x^2 + yz \, ds = \int_0^2 (t^6 + e^t \sin t) \cdot \sqrt{9t^4 + e^{2t} + \cos^2 t} \; dt$

$\vec{r}'(t) = (3t^2, e^t, \cos t)$

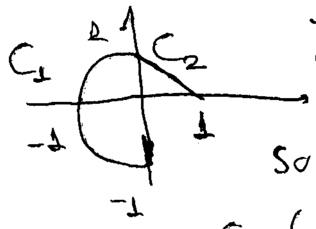
$|\bar{r}'(t)| = \sqrt{(3t^2)^2 + (e^t)^2 + \cos^2 t} \qquad ds = |\bar{r}'(t)| \, dt$

$C_2: (x,y) = (\arctan(t), \ln(t)), \; 2 \le t \le 5$

$\displaystyle\int_{C_2} \frac{y}{x} \, ds = \int_2^5 \frac{\ln t}{\arctan(t)} \sqrt{\left(\frac{1}{1+t^2}\right)^2 + \left(\frac{1}{t}\right)^2} \; dt$

$\bar{r}'(t) = \left(\frac{1}{1+t^2}, \frac{1}{t}\right), \; |\bar{r}'(t)| = \sqrt{\left(\frac{1}{1+t^2}\right)^2 + \left(\frac{1}{t}\right)^2}$

Example




$\displaystyle\int_C xy \, ds = ?$ where $C = C_1 \cup C_2$

Solution: First parametrize $C_1$ and $C_2$:

$C_1: (x,y) = (\cos t, \sin t), \; \frac{\pi}{2} \le t \le \frac{3\pi}{2}$

$C_2: (x,y) = (t, 1-t), \; 0 \le t \le 1$

For $C_1$, $ds = |\bar{r}'(t)| \, dt = |(-\sin t, \cos t)| \, dt = \sqrt{(-\sin t)^2 + (\cos t)^2} \, dt = 1 \, dt$

For $C_2$, $ds = |\bar{r}'(t)| \, dt = |(1,-1)| \, dt = \sqrt{1^2 + (-1)^2} \, dt = \sqrt{2} \, dt$

$\displaystyle\int_C xy \, ds = \int_{C_1} xy \, ds + \int_{C_2} xy \, ds$

$\displaystyle = \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \cos t \cdot \sin t \cdot 1 \, dt + \int_0^1 t(1-t) \cdot \sqrt{2} \, dt = \_\_\_\_$

Interpretation and applications of the line integral $\int_C f(x,y,z)\,ds$

1) If $f(x,y,z)$ is the linear density (gr/unit length) of a wire in the shape of the curve $C$, then total mass $M$ of the wire is approximately $M \approx \sum_{i=1}^{n} f(x_i,y_i,z_i)\Delta s_i$

where $\Delta s_i$ = length of $i^{th}$ arc when $C$ is divided into $n$ small arcs

In the limit as we take smaller arcs as $n \to \infty$, we get

$$M = \int_C f(x,y,z)\,ds$$

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2) Center of mass of a wire in the shape of a curve $C$ with density (linear density in gr/unit length units) function $f(x,y,z)$

If center of mass is the point $P(x_0, y_0, z_0)$, then

$$x_0 = \frac{\int_C x \cdot f(x,y,z)\,ds}{M} \qquad y_0 = \frac{\int_C y \cdot f(x,y,z)\,ds}{M} \quad , z_0 = \frac{\int_C z \cdot f(x,y,z)\,ds}{M}$$

where $M = \int_C f(x,y,z)\,ds$ is the total mass of the wire.

Note: If the wire is made of homogeneous material, then $f(x,y,z)$, the density, will be a constant function.

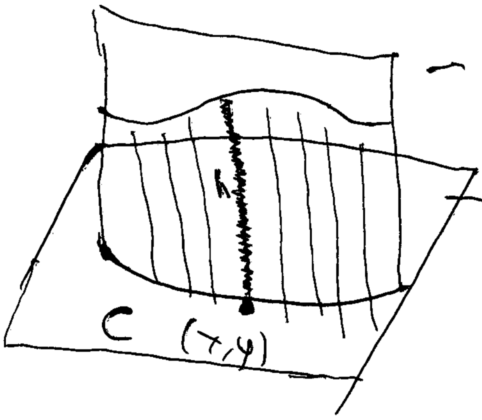
Note 2: Similar formulas hold for a curve $C$ in $\mathbb{R}^2$ where $f(x,y)$ is the linear density function.

# Applications of Line Integrals of functions

## 3) Surface area of a wall

For a wall (vertical wall) which projects to a plane curve $C$ (on a horizontal plane), surface area of the wall is

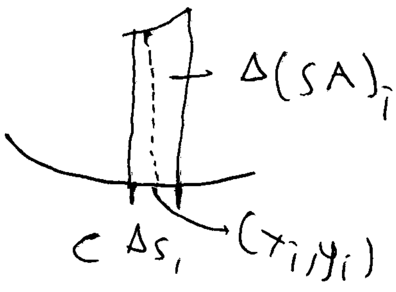
$$S.A. = \int_C h(x,y)\, ds$$ where $h(x,y)$ is the height of the wall over the point $(x,y) \in C$.




— Cilinder with basecurve C

→ horizontal plane (xy-plane)

The area of the wall (shaded surface) is $\int_C h(x,y)\,ds$




$$\Delta(SA)_i \approx h(x_i,y_i)\cdot \Delta s_i \Rightarrow SA \approx \sum_{i=1}^{n} h(x_i,y_i)\,\Delta s_i$$

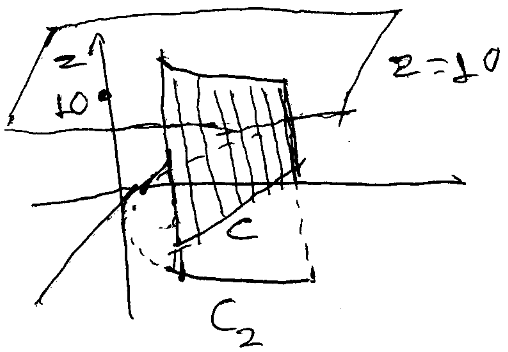
$$SA = \int_C h(x,y)\,ds$$

Example

A wall is constructed on a hill such that the base curve of the wall is parametrized as $C:(x,y,z)=(t,t^2,t^3),\ 1\le t\le 2$

If the top of the wall is horizontal and lies on $z=10$ plane, calculate the surface area of the wall (surface area of one side only)

Solution:




Projection of the base curve $C$ to $xy$-plane:

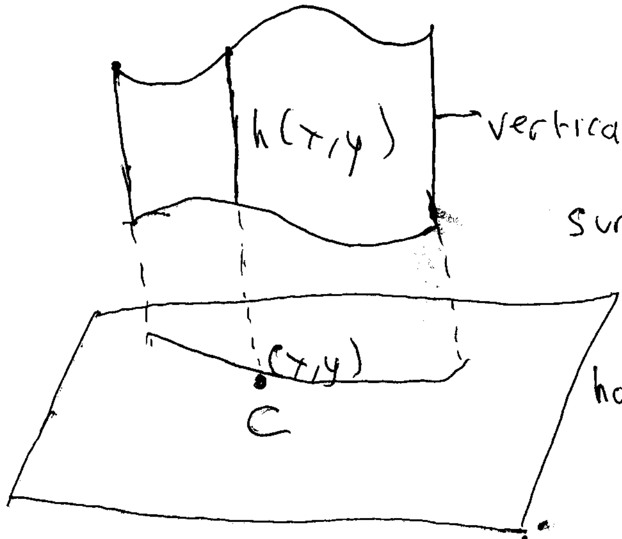
$C_2:(x,y)=(t,t^2),\ 1\le t\le 2$

height of the wall over $(t,t^2)\in C_2$:

$h=10-t^3$ $\quad$ ($h(x,y)=10-z=10-t^3$ at $(x,y)=(t,t^2)$)

Then $S.A=\int_{C_2} h(x,y)\,ds=\int_1^2 (10-t^3)\cdot\sqrt{1+4t^2}\,dt=\ldots$

for $C_2$, $ds=|\vec{r}\,'(t)|\,dt=|(1,2t)|\,dt=\sqrt{1+4t^2}\,dt$




Surface Area: $S.A=\int_C h(x,y)\,ds$

★ height should be integrated on the curve which is projection of the vertical wall to a horizontal plane.