11.1 Vector Functions of One Variable

A vector function $\vec{r}$ of one variable t in $\mathbb{R}^3$ is given by

$$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle, \quad a \le t \le b$$
$$= x(t)\cdot\vec{i} + y(t)\cdot\vec{j} + z(t)\cdot\vec{k}, \quad a \le t \le b$$
$$= (x(t), y(t), z(t)), \quad a \le t \le b$$

$\vec{r}(t)$ is a vector in $\mathbb{R}^3$ which depends on the variable t.

Components of $\vec{r}(t)$ are also functions of t which are $x(t), y(t), z(t)$.

Domain of $\vec{r}$ is given as the interval $[a,b]$ above, but in general domain of $\vec{r}$ can be any subset D of $\mathbb{R}$.

$$\vec{r} : D \to \mathbb{R}^3$$
$$\vec{r}(t) = (x(t), y(t), z(t)) \quad \text{for any } t \in D \quad (D \subseteq \mathbb{R})$$

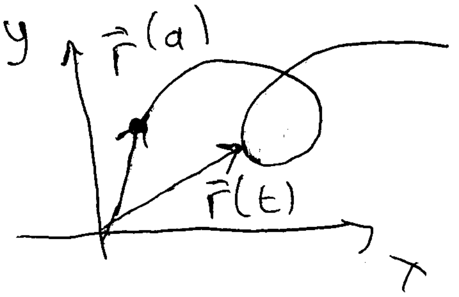
A vector function of one variable in $\mathbb{R}^2$ has two components

$$\vec{r} : D \to \mathbb{R}^2 \quad (D \subseteq \mathbb{R})$$
$$\vec{r}(t) = (x(t), y(t)) \quad \text{for all } t \in D.$$

One interpretation of such a function $\vec{r}$ is:

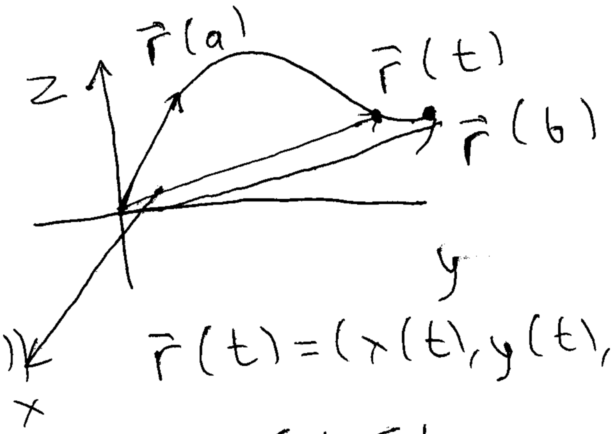
$\vec{r}(t)$ is the position of a moving point P in space ($\mathbb{R}^3$ or $\mathbb{R}^2$) at time t, where we consider its position (x,y,z) as a vector from origin O to P, $\vec{P} = \vec{OP} = \vec{r}(t)$

$$= (x(t), y(t), z(t))$$



$\vec{r}(t) = (x(t), y(t))$

$a \leq t \leq b$



$\vec{r}(t) = (x(t), y(t), z(t))$

$a \leq t \leq b$

Limits and Continuity

For $\vec{r}(t) = (x(t), y(t), z(t))$, $t \in D$ $(D \subseteq \mathbb{R})$

$$\lim_{t \to t_0} \vec{r}(t) = \lim_{t \to t_0} (x(t), y(t), z(t))$$

$$= \left(\lim_{t \to t_0} x(t), \lim_{t \to t_0} y(t), \lim_{t \to t_0} z(t) \right)$$

if all 3 limits exist.

$\vec{r}(t)$ is called continuous at $t = t_0$ if $\lim_{t \to t_0} \vec{r}(t) = \vec{r}(t_0)$

- Limits are calculated componentwise.
- $\vec{r}(t)$ is continuous iff all component functions $x(t)$, $y(t)$ and $z(t)$ are continuous.

<u>Derivatives</u> of vector valued functions of one variable

For $\bar{r}(t) = (x(t), y(t), z(t))$, $a \le t \le b$,

$$\vec{r}'(t_0) = \frac{d}{dt}\bar{r}(t)\Big|_{t=t_0}$$

$$= \lim_{h \to 0} \frac{\bar{r}(t_0+h) - \bar{r}(t_0)}{h}$$

$$= \lim_{h \to 0} \left(\frac{x(t_0+h)-x(t_0)}{h}, \frac{y(t_0+h)-y(t_0)}{h}, \frac{z(t_0+h)-z(t_0)}{h} \right)$$

$$= (x'(t_0), y'(t_0), z'(t_0))$$ if all 3 derivatives exist.

In short:

$$\vec{r}'(t) = (x'(t), y'(t), z'(t))$$

Interpretation: If $\bar{r}(t)$ is considered as the position of a moving point at time t, then

$\vec{r}'(t) = \bar{v}(t)$ is velocity at time t,

$\vec{r}''(t) = (\bar{r}'(t))' = \vec{v}'(t) = \bar{a}(t)$ is the acceleration at time t.

$$|\bar{r}'(t)| = |(x'(t), y'(t), z'(t))|$$

$$= \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}$$

$|\bar{r}'(t)|$: Speed at time t (a scalar function)

- Note: For vector functions of one variable in $\mathbb{R}^2$, similar formulas are valid using 2 component functions.

Integrals of vector valued functions of one variable

For $\vec{r}(t)=(x(t),y(t),z(t))$

Indefinite Integral:

$\int \vec{r}(t)dt=\vec{F}(t)+\vec{V}$ where $\vec{F}'(t)=\vec{r}(t)$ and $\vec{V}$ is a constant vector.

$\vec{F}'(t)=\vec{r}(t) \Rightarrow \vec{F}$ is called an antiderivative of $\vec{r}$

$\vec{F}(t)=(P(t),Q(t),R(t)) \Rightarrow \vec{F}'(t)=\vec{r}(t)$

$$(P'(t),Q'(t),R'(t))=(x(t),y(t),z(t))$$

$$\begin{aligned}\int \vec{r}(t)dt &= \int (x(t),y(t),z(t))dt \\ &= \left(\int x(t)dt, \int y(t)dt, \int z(t)dt\right) \\ &= (P(t)+C_1, Q(t)+C_2, R(t)+C_3) \\ &= (P(t),Q(t),R(t))+(C_1,C_2,C_3) \\ &= \vec{F}(t)+\vec{V}\end{aligned}$$

where $\vec{V}=(C_1,C_2,C_3)$ is an arbitrary constant vector.

Definite Integral:

$$\begin{aligned}\int_a^b \vec{r}(t)dt &= \int_a^b (x(t),y(t),z(t))dt \\ &= \left(\int_a^b x(t)dt, \int_a^b y(t)dt, \int_a^b z(t)dt\right) \\ &= \vec{F}(b)-\vec{F}(a)\end{aligned}$$

where $\vec{F}'(t)=\vec{r}(t)$

★ Net Change:

$$\underbrace{\vec{F}(b)-\vec{F}(a)}_{\text{Net Change in } \vec{F}(t) \text{ on the interval } [a,b]}=\int_a^b \vec{F}'(t)dt$$

Net Change in $\vec{F}(t)$ on the interval $[a,b]$

Example, Displacement: $\vec{r}(b)-\vec{r}(a)=\int_a^b \vec{v}(t)dt$

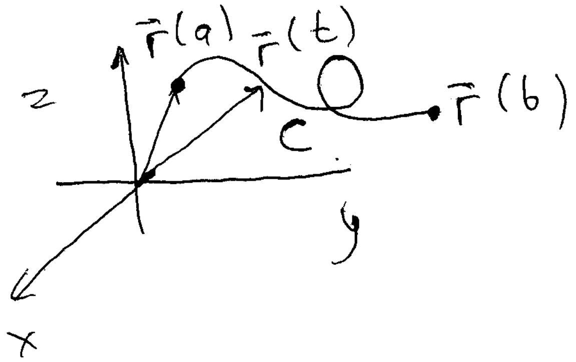
where $\vec{r}$: position, $\vec{v}=\vec{r}'$: velocity function.

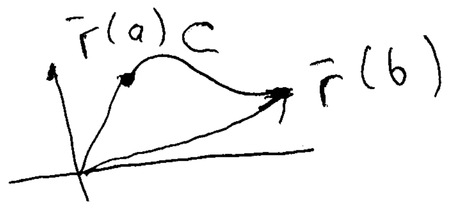
11.3 | Curves and Parametrizations

A curve C is the image of a vector valued function of one variable $\vec{r}(t)$ defined on an interval.

$$C = \{ (x,y,z) \in \mathbb{R}^3 \mid (x,y,z) = \vec{r}(t) \text{ for some } t \}$$

We say that C is the curve parametrized by $\vec{r}(t)$.
Considering $\vec{r}(t)$ as position of a moving point at time t, C is the path of the motion. C is the set of all points in space through which this moving point passes during that motion.





$C: \bar{r}(t) = (x(t), y(t)),\ a \le t \le b$

$C: \bar{r}(t) = (x(t), y(t), z(t)),\ a \le t \le b$

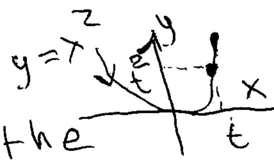
$\bar{r}(a)$: initial point of C

$\bar{r}(b)$: terminal point of C

- If component functions are continuous, $\bar{r}(t)$ is called a continuous parametrization, and the curve C has no breaks (C is connected).

Examples

1) $\bar{r}(t) = (t, t^2),\ t \in \mathbb{R}$ parametrizes the parabola $y = x^2$.



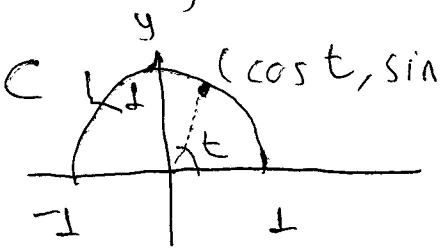
$\bar{r}(t) = (x,y) = (t, t^2) \Rightarrow x = t$, $y = t^2$

$\Rightarrow y = t^2 = x^2 \Rightarrow y = x^2$ for all $(x,y) = \bar{r}(t)$.

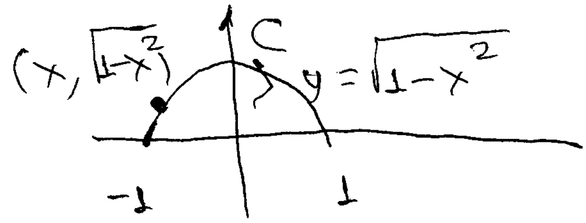
And for any (x,y) on graph of $y = x^2$, $(x,y) = (x, x^2) = \bar{r}(x)$

2) The two parametrizations below both parametrize the same curve C (the upper semicircle), but in different directions:

$$\vec{r}_1(t) = (x,y) = (\cos(t), \sin(t)), \quad 0 \leq t \leq \pi$$

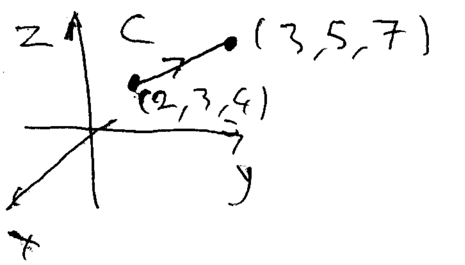


$$\vec{r}_2(x) = (x,y) = (x, \sqrt{1-x^2}), \quad -1 \leq x \leq 1$$



* The same curve C can indeed be parametrized in infinitely many different ways.

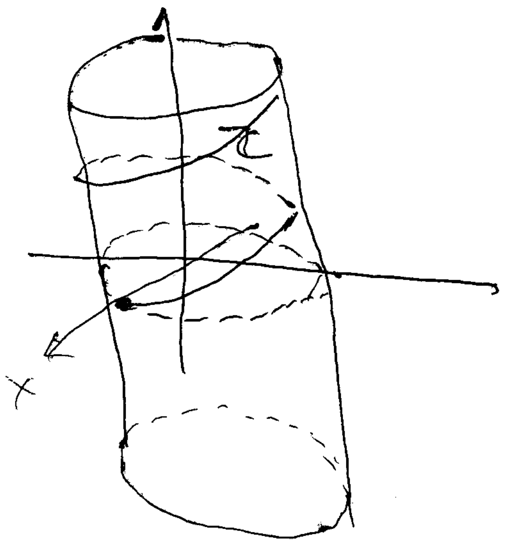
3) $\vec{r}(t) = (x,y,z) = (1+t, 1+2t, 1+3t), \quad 1 \leq t \leq 2$

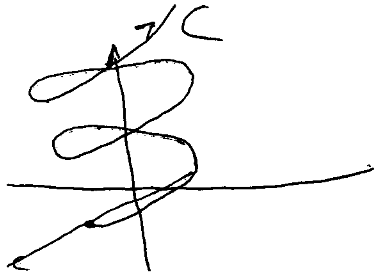


For $t \in \mathbb{R}$, the above is actually the vector equation of the line L through $(1,1,1)$ with direction vector $\vec{v} = (1,2,3)$. Since $1 \leq t \leq 2$, $\vec{r}(t)$ parametrizes the line segment from $\vec{r}(1) = (2,3,4)$ to $\vec{r}(2) = (3,5,7)$.

4) Helix:

C: $\bar{r}(t) = (\cos t, \sin t, t)$, $0 \le t$

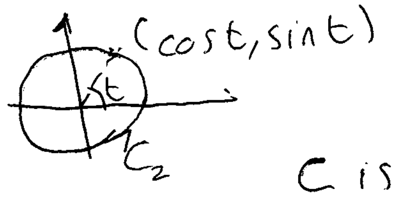




Projection of C to xy-plane has parametrization

$C_2: \bar{r}_2(t) = (x, y) = (\cos t, \sin t)$, $0 \le t \Rightarrow C_2$ is a circle

(projection of (x, y, z) to xy-plane is $(x, y, 0)$ which is (x, y) in $\mathbb{R}^2$)



As the projection of $\bar{r}(t)$ to xy ~~point~~ plane is at the point $(x, y) = (\cos t, \sin t)$ on C_2, z coordinate of $\bar{r}(t)$ on C is t.

C is the helix which winds up the cylinder $x^2 + y^2 = 1$

(Note that C is on the cilinder $x^2 + y^2 = 1$ in $\mathbb{R}^3$ since

$(x, y, z) \in C \Rightarrow x = \cos t, y = \sin t, z = t$, hence

$$x^2 + y^2 = (\sin t)^2 + (\cos t)^2 = \sin^2 t + \cos^2 t = 1$$

$$\Rightarrow x^2 + y^2 = 1.$$

Every point $(x, y, z) \in C$ satisfies the equation of the cilinder, so C lies on the cilinder.)

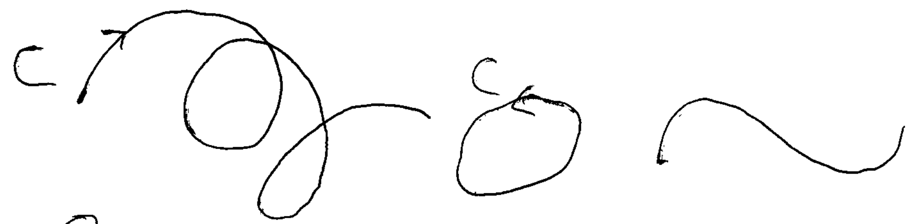
Smooth Parametrization and smooth curve

A parametrization $\vec{r}(t)$, $a \le t \le b$ of a curve C is called a smooth parametrization if $\vec{r}'(t)$ is continuous on (a,b) AND $\vec{r}'(t) \neq \vec{0}$ for any t.

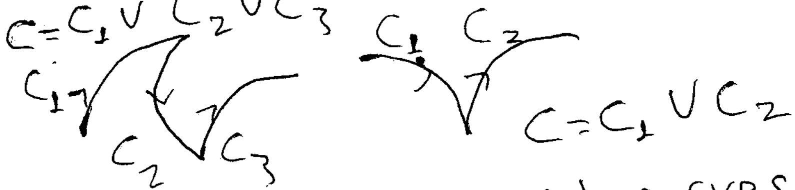
With the interpretation of $\vec{r}(t)$ being the position of a moving point on C at time t, this means that the velocity $\vec{v}(t) = \vec{r}'(t)$ changes continuously and $\vec{v}$ never becomes $\vec{0}$ (point never stops)

C is called a smooth curve if it has a smooth parametrization

C is called a piecewise smooth curve if C is a continuous curve which is a union of finitely many smooth curves

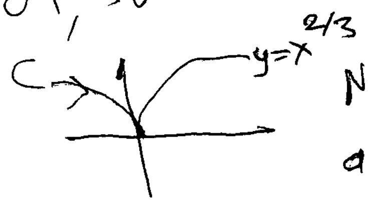


C smooth curves



C piecewise smooth curves

Example: $\vec{r}(t) = (t^3, t^2)$, $t \in \mathbb{R}$ parametrizes the curve $y = x^{2/3}$. $\vec{r}'(t) = (3t^2, 2t)$ is continuous but $\vec{r}'(0) = (0,0)$, so this is not a smooth parametrization.



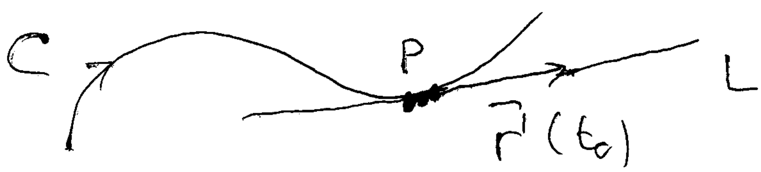
Note that C is singular at $(0,0)$ (when $t=0$)

there is no tangent line to C at $(0,0)$.

For C given by a parametrization $\vec{r}(t)$,
if $\vec{r}'(t_0)$ is not $\vec{0}$, then
Tangent Line to C at the point $P = \vec{r}(t_0)$ is:

$$L: (x,y,z) = \vec{r}(t_0) + t \cdot \vec{r}'(t_0), \ t \in \mathbb{R}$$

Here $\vec{r}'(t_0) = \vec{v}(t_0)$ is a direction vector of the tangent line.



$\vec{P} = \vec{r}(t_0)$

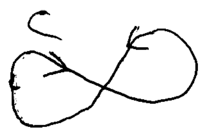
A smooth parametrization on a curve C determines a direction on C from initial point to terminal point.

$\vec{r}'(t) = \vec{v}(t)$: velocity vector on C
: tangent vector to C

} shows the direction on C at the point $\vec{r}(t)$.

Closed Curve: $\vec{r}(a) = \vec{r}(b)$ (initial point = terminal point)

simple closed curve



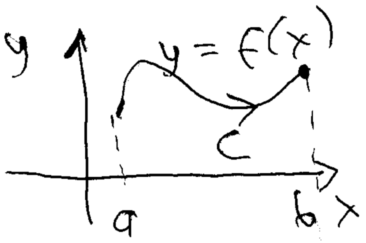
closed but not simple
simple curve means there is no self crossing.

clockwise oriented simple closed curve

counter clockwise oriented simple closed curve.

Parametrizing Curves

1) Graph of a function $y = f(x)$ on $[a, b]$

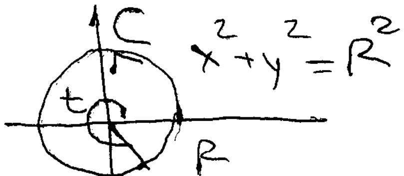


C is parametrized as:

$$C : \bar{r}(x) = (x, y) = (x, f(x)), \quad a \leq x \leq b$$

Here x is the parameter.

2) Circles



$(x, y) = (R\cos t, R\sin t)$

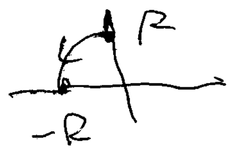
The circle C is parametrized as:

$$C : (x, y) = \bar{r}(t) = (R\cos t, R\sin t)$$
$$0 \leq t \leq 2\pi$$

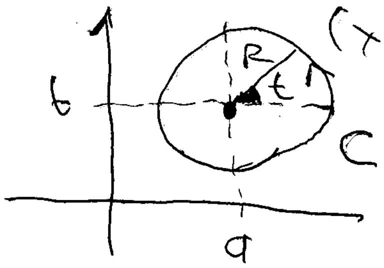
Direction on C is counterclockwise.

$0 \leq t \leq \pi \Rightarrow$ parametrizes upper semicircle

$\frac{\pi}{2} \leq t \leq \pi \Rightarrow$ parametrizes



$0 \leq t \leq 4\pi \Rightarrow$ parametrization traverses the circle twice.



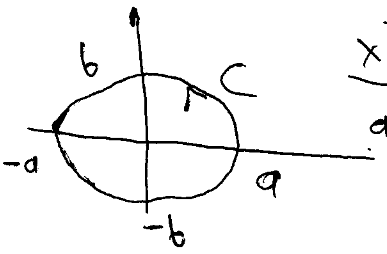
$(x, y) = (a + R\cos t, b + R\sin t)$

C → A parametrization is:

$$C : (x, y) = (a + R\cos t, b + R\sin t),$$
$$0 \leq t \leq 2\pi$$

Parametrizing Curves

3) Ellipses



$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

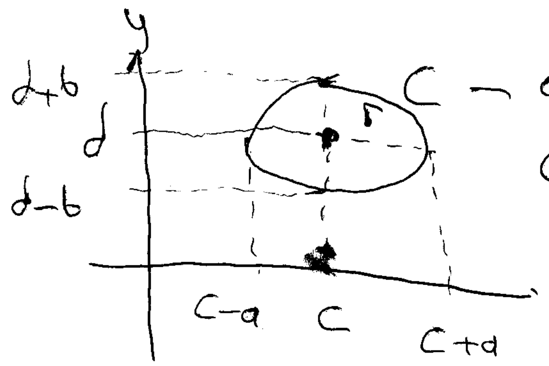
Parametrization of C:

$(x,y)=\vec{r}(t)=(a\cos t,\ b\sin t),\ 0\le t\le 2\pi$

$$\left(x=a\cos t,\ y=b\sin t \Rightarrow \frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{a^2\cos^2 t}{a^2}+\frac{b^2\sin^2 t}{b^2}=\cos^2 t+\sin^2 t=1\right)$$

Shifted Ellipse:

$$\frac{(x-c)^2}{a^2}+\frac{(y-d)^2}{b^2}=1$$



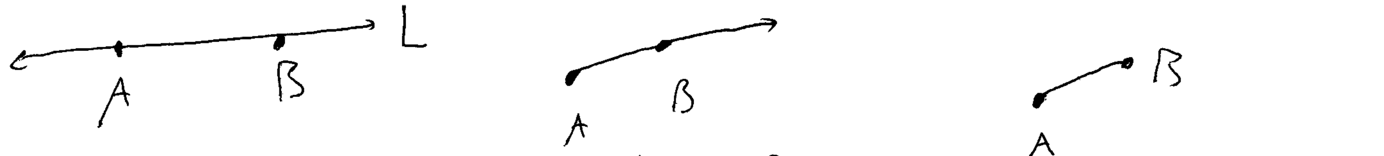
C — a parametrization is:

$C: (x,y)=(c+a\cos t,\ d+b\sin t),$

$0\le t\le 2\pi$

Parametrizing Curves

4) Lines, Rays, Line Segments (in $\mathbb{R}^2$ or $\mathbb{R}^3$)



The ray $[AB$

Line segment $[AB]$

For two points A and B,

let $\vec{v} = \overrightarrow{AB}$

Parametrization of

the line L: $\vec{r}(t) = \vec{A} + t\vec{v}$, $t \in \mathbb{R}$

the ray $[AB$: $\vec{r}(t) = \vec{A} + t\vec{v}$, $t \geq 0$

the line segment $[AB]$: $\vec{r}(t) = \vec{A} + t\vec{v}$, $0 \leq t \leq 1$.

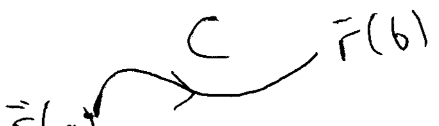
5) Reversing the direction of a parametrization:

For C: $\vec{r}(t) = (x(t), y(t), z(t))$, $a \leq t \leq b$

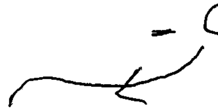
$\vec{r}_2(t) = \vec{r}(-t)$, $-b \leq t \leq -a$ is the parametrization of C with reversed direction.

$-C$: $\vec{r}_2(t) = (x(-t), y(-t), z(-t))$

$-b \leq t \leq -a$



$-C$: same curve C with reversed direction

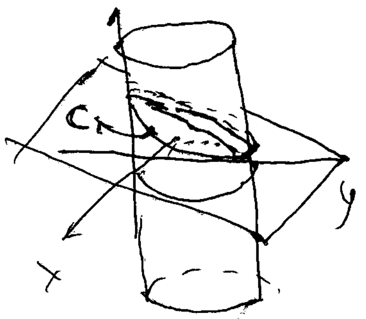


Parametrizing Curves

6) Intersection curve of two surfaces

Example:
Parametrize the curve C which is the intersection of the two surfaces $x+2y+3z=6$ and $x^2+(y-1)^2=1$ in $\mathbb{R}^3$.

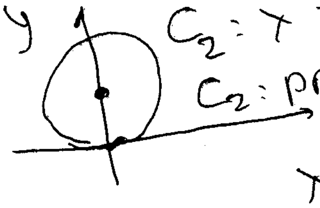
Solution:



$x^2+(y-1)^2=1$ is a cylinder.

$x+2y+3z=6$ is a plane.

C lies on the cillinder, so projection of C to xy-plane is the circle $x^2+(y-1)^2=1$



$C_2: x^2+(y-1)^2=1$

C_2: projection of C to xy-plane.

Parametrization of C_2:

$(x,y) \in C_2$

$\Rightarrow (x,y) = (\cos t, 1+\sin t)$ $0 \leq t \leq 2\pi$

$(x,y,z) \in C \Rightarrow (x,y) \in C_2$

$\Rightarrow x=\cos t,\ y=1+\sin t,\ 0\leq t\leq 2\pi$

$x+2y+3z=6 \Rightarrow z=\dfrac{6-x-2y}{3} = \dfrac{6-\cos t-2-2\sin t}{3}$

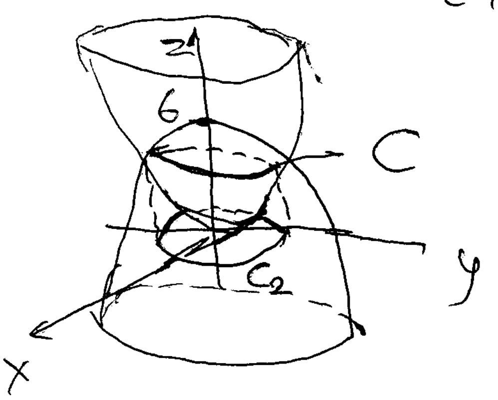
Thus

$C: (x,y,z) = \left(\cos t, 1+\sin t, \dfrac{4-\cos t-2\sin t}{3}\right),\ 0\leq t\leq 2\pi$

Example

Parametrize the intersection of the paraboloids $z = x^2 + 2y^2$ and $z = 6 - x^2 - y^2$.

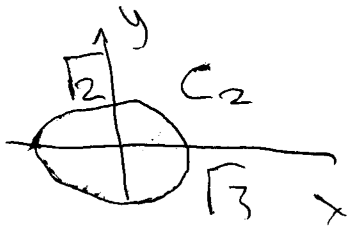
Solution:



$$(x,y,z) \in C \iff z = x^2 + 2y^2$$
$$z = 6 - x^2 - y^2$$

Eliminating z using the 2 equations,

$$x^2 + 2y^2 = 6 - x^2 - y^2$$
$$2x^2 + 3y^2 = 6$$
$$\frac{x^2}{3} + \frac{y^2}{2} = 1 -$$



C_2: projection of C to xy-plane.

$C_2 : (x,y) = (\sqrt{3}\cos t, \sqrt{2}\sin t), \ 0 \le t \le 2\pi$

Then $(x,y,z) \in C \Rightarrow (x,y) \in C_2 \Rightarrow x = \sqrt{3}\cos t, \ y = \sqrt{2}\sin t$

$z = x^2 + 2y^2 \Rightarrow z = 3\cos^2 t + 4\sin^2 t = 3 + \sin^2 t$

$C : (x,y,z) = (\sqrt{3}\cos t, \sqrt{2}\sin t, 3 + \sin^2 t), \ 0 \le t \le 2\pi$

Warning: Above projection of C to xy-plane is all of C_2 as we can understand from the graph of the paraboloids.
If we eliminated x from the 2 equations, we would get $C_3 : z = 3 + \frac{y^2}{2}$. Projection of C to yz plane is in C_3, but not all of C_3!!!