### 11.4 Vector Functions of One Variable

A vector function \( \vec{r} \) of one variable \( t \) in \( \mathbb{R}^3 \) is given by

\[
\vec{r}(t) = \langle x(t), y(t), z(t) \rangle, \quad a \leq t \leq b
\]

\[
= x(t) \cdot \vec{i} + y(t) \cdot \vec{j} + z(t) \cdot \vec{k}, \quad a \leq t \leq b
\]

\[
= (x(t), y(t), z(t)), \quad a \leq t \leq b
\]

\( \vec{r}(t) \) is a vector in \( \mathbb{R}^3 \) which depends on the variable \( t \).

Components of \( \vec{r}(t) \) are also functions of \( t \) which are \( x(t), y(t), z(t) \).

Domain of \( \vec{r} \) is given as the interval \([a, b]\) above, but in general domain of \( \vec{r} \) can be any subset \( D \) of \( \mathbb{R} \).

\[
\vec{r} : D \rightarrow \mathbb{R}^3
\]

\[
\vec{r}(t) = (x(t), y(t), z(t)) \quad \text{for any } t \in D
\]

A vector function of one variable in \( \mathbb{R}^2 \) has two components

\[
\vec{r} : D \rightarrow \mathbb{R}^2 \quad (D \subseteq \mathbb{R})
\]

\[
\vec{r}(t) = (x(t), y(t)) \quad \text{for all } t \in D.
\]

One interpretation of such a function \( \vec{r} \) is:

\( \vec{r}(t) \) is the position of a moving point \( P \) in space \( (\mathbb{R}^3 \text{ or } \mathbb{R}^2) \) at time \( t \), where we consider its position \( (x, y, z) \) as a vector

from origin \( O \) to \( P \),

\[
\vec{P} = \overrightarrow{OP} = \vec{r}(t)
\]

\[
= (x(t), y(t), z(t))
\]
\[ \vec{r}(t) = (\gamma(t), y(t), z(t)) \quad \text{for} \quad a \leq t \leq b. \]

**Limits and Continuity**

For \( \vec{r}(t) = (\gamma(t), y(t), z(t)) \), \( t \in D \quad (D \subseteq \mathbb{R}) \)

\[ \lim_{t \to t_0} \vec{r}(t) = \lim_{t \to t_0} (\gamma(t), y(t), z(t)) = (\lim_{t \to t_0} \gamma(t), \lim_{t \to t_0} y(t), \lim_{t \to t_0} z(t)) \]

if all 3 limits exist.

\( \vec{r}(t) \) is called **continuous** at \( t = t_0 \) if \( \lim_{t \to t_0} \vec{r}(t) = \vec{r}(t_0) \)

- Limits are calculated componentwise.
- \( \vec{r}(t) \) is continuous iff all component functions \( \gamma(t), y(t) \) and \( z(t) \) are continuous.
Derivatives of vector valued functions of one variable

For \( \mathbf{r}(t) = (x(t), y(t), z(t)) \), \( a \leq t \leq b \),

\[
\mathbf{r}'(t_0) = \frac{d}{dt} \mathbf{r}(t) \bigg|_{t=t_0} = \lim_{h \to 0} \frac{\mathbf{r}(t_0+h) - \mathbf{r}(t_0)}{h} \\
= \lim_{h \to 0} \left( \frac{x(t_0+h)-x(t_0)}{h}, \frac{y(t_0+h)-y(t_0)}{h}, \frac{z(t_0+h)-z(t_0)}{h} \right) \\
= (x'(t_0), y'(t_0), z'(t_0)) \text{ if all 3 derivatives exist.}
\]

In short,

\[
\mathbf{r}'(t) = (x'(t), y'(t), z'(t))
\]

Interpretation: If \( \mathbf{r}(t) \) is considered as the position of a moving point at time \( t \), then

\[
\mathbf{r}'(t) = \mathbf{v}(t) \text{ is velocity at time } t,
\]

\[
\mathbf{r}''(t) = \mathbf{a}(t) \text{ is the acceleration at time } t.
\]

\[
|\mathbf{r}'(t)| = \left| (x'(t), y'(t), z'(t)) \right| = \sqrt{\left(x'(t)\right)^2 + \left(y'(t)\right)^2 + \left(z'(t)\right)^2}
\]

\[
|\mathbf{r}'(t)| \text{ is speed at time } t \text{ (a scalar function)}
\]

Note: For vector functions of one variable in \( \mathbb{R}^2 \), similar formulas are valid using 2 component functions.
Integrals of vector valued functions of one variable

For \( \mathbf{r}(t) = (x(t), y(t), z(t)) \)

Indefinite Integral:
\[
\int \mathbf{r}(t) \, dt = \mathbf{F}(t) + \mathbf{V}
\]
where \( \mathbf{F}'(t) = \mathbf{r}(t) \) and \( \mathbf{V} \) is a constant vector.

\( \mathbf{F}'(t) = \mathbf{r}(t) \Rightarrow \mathbf{F} \) is called an antiderivative of \( \mathbf{r} \)

\( \mathbf{F}(t) = (P(t), Q(t), R(t)) \Rightarrow \mathbf{F}'(t) = \mathbf{r}(t) \)

\( (P'(t), Q'(t), R'(t)) = (x(t), y(t), z(t)) \)

\[
\int \mathbf{r}(t) \, dt = \int (x(t), y(t), z(t)) \, dt
\]

\[
= (\int x(t) \, dt, \int y(t) \, dt, \int z(t) \, dt)
\]

\[
= (P(t) + C_1, Q(t) + C_2, R(t) + C_3)
\]

\[
= (P(t), Q(t), R(t)) + (C_1, C_2, C_3)
\]

\[
= \mathbf{F}(t) + \mathbf{V}
\]
where \( \mathbf{V} = (C_1, C_2, C_3) \) is an arbitrary constant vector.

Definite Integral:
\[
\int_{a}^{b} \mathbf{r}(t) \, dt = \int_{a}^{b} (x(t), y(t), z(t)) \, dt
\]

\[
= (\int_{a}^{b} x(t) \, dt, \int_{a}^{b} y(t) \, dt, \int_{a}^{b} z(t) \, dt)
\]

\[
= \mathbf{F}(b) - \mathbf{F}(a) \text{ where } \mathbf{F}'(t) = \mathbf{r}(t)
\]

Net Change:
\[
\mathbf{F}(b) - \mathbf{F}(a) = \int_{a}^{b} \mathbf{r}(t) \, dt
\]

Net change in \( \mathbf{F}(t) \) on the interval \([a, b] \)

Example: Displacement:
\[
\mathbf{F}(b) - \mathbf{F}(a) = \int_{a}^{b} \mathbf{v}(t) \, dt
\]
where \( \mathbf{F} \) : position, \( \mathbf{v} = \mathbf{F}' \) : velocity function.
A curve $C$ is the image of a vector valued function of one variable $\mathbf{r}(t)$ defined on an interval.

$$C = \{ (x, y, z) \in \mathbb{R}^3 \mid (x, y, z) = \mathbf{r}(t) \text{ for some } t \}$$

We say that $C$ is the curve parametrized by $\mathbf{r}(t)$. Considering $\mathbf{r}(t)$ as position of a moving point at time $t$, $C$ is the path of the motion. $C$ is the set of all points in space through which this moving point passes during that motion.

\[ \mathbf{r}(a) \quad \mathbf{r}(t) \quad \mathbf{r}(b) \]

\[ \mathbf{r}(a): \text{initial point of } C \]
\[ \mathbf{r}(b): \text{terminal point of } C \]

If component functions are continuous, $\mathbf{r}(t)$ is called a continuous parametrization, and the curve $C$ has no breaks (C is connected).

Examples

1) $\mathbf{r}(t) = (t, t^2)$, $t \in \mathbb{R}$ parametrizes the parabola $y = x^2$. $\mathbf{r}(t) = (x, y) = (t, t^2) \implies x = t, y = t^2$

$y = t^2 = x^2 \implies y = x^2$ for all $(x, y) = \mathbf{r}(t)$.

And for any $(x, y)$ on graph of $y = x^2$, $(x, y) = (x, x^2) = \mathbf{r}(t)$.
2) The two parametrizations below both parametrize the same curve \( C \) (the upper semicircle), but in different directions:

\[
\vec{r}_1(t) = (x, y) = (\cos(t), \sin(t)), \quad 0 \leq t \leq \pi
\]

\[
\vec{r}_2(x) = (x, y) = (x, \sqrt{1-x^2}), \quad -1 \leq x \leq 1
\]

*The same curve \( C \) can indeed be parametrized in infinitely many different ways.*

3) \( \vec{r}(t) = (x, y, z) = (1 + t, 1 + 2t, 1 + 3t), \quad 1 \leq t \leq 2 \)

\( \vec{r}(1) = (2, 3, 4) \) \( \vec{r}(2) = (3, 5, 7) \) For \( t \in [1, 2] \), the above is actually the vector equation of the line \( L \) through \((1,1,1)\) with direction vector \( \vec{v} = (1, 2, 3) \). Since \( 1 \leq t \leq 2 \), \( \vec{r}(t) \) parametrizes the line segment from \( \vec{r}(1) = (2, 3, 4) \) to \( \vec{r}(2) = (3, 5, 7) \).
4) Helix: 
\[ C: \vec{r}(t) = (\cos t, \sin t, t), \quad 0 \leq t \]

Projection of \( C \) to \( xy \)-plane has parametrization
\[ C_2: \vec{r}_2(t) = (x, y) = (\cos t, \sin t), \quad 0 \leq t \Rightarrow C_2 \text{ is a circle} \]
(projection of \((x, y, z)\) to \(xy\)-plane is \((x, y, 0)\) which is \((x, y)\) in \(\mathbb{R}^2\) as the projection of \(\vec{r}(t)\) to \(xy\) plane
\[ (\cos t, \sin t) \text{ is at the point } (x, y) = (\cos t, \sin t) \text{ on } C_2, \text{ z coordinate of } \vec{r}(t) \text{ on } C \text{ is } t. \]

\( C \), the helix which winds up the cylinder
\[ x^2 + y^2 = 1 \]
(Note that \( C \) is on the cylinder \( x^2 + y^2 = 1 \) in \(\mathbb{R}^3 \) since \((x, y, z) \in C \Rightarrow x = \cos t, y = \sin t, z = t \), hence
\[ z^2 + y^2 = (\sin t)^2 + (\cos t)^2 = \sin^2 t + \cos^2 t = 1 \]
\[ \Rightarrow x^2 + y^2 = 1. \]
Every point \((x, y, z) \in C\) satisfies the equation of the cylinder, so \( C \) lies on the cylinder.)
Smooth Parametrization and Smooth Curve

A parametrization \( \vec{r}(t), a \leq t \leq b \) of a curve \( C \) is called a smooth parametrization if \( \vec{r}(t) \) is continuous on \((a,b)\) AND \( \vec{r}'(t) \neq \vec{0} \) for any \( t \).

With the interpretation of \( \vec{r}(t) \) being the position of a moving point on \( C \) at time \( t \), this means that the velocity \( \vec{v}(t) = \vec{r}'(t) \) changes continuously and \( \vec{v} \) never becomes \( \vec{0} \) (point never stops).

\( C \) is called a smooth curve if it has a smooth parametrization.

\( C \) is called a piecewise smooth curve if \( C \) is a continuous curve which is a union of finitely many smooth curves.

\[ C \]

\[ C = C_1 \cup C_2 \cup C_3 \]

\[ C = C_1 \cup C_2 \]

\[ C \text{ piecewise smooth curves} \]

Example: \( \vec{r}(t) = (t^3, t^2), t \in \mathbb{R} \) parametrizes the curve \( y = x^{2/3} \). \( \vec{r}(t) = (t^3, t^2) \) is continuous but \( \vec{r}'(0) = (0, 0) \), so this is not a smooth parametrization. \( C \) is singular at \((0,0)\) (when \( t = 0 \)).

Note that \( C \) is not smooth at \((0,0)\) (when \( t = 0 \)). There is no tangent line to \( C \) at \((0,0)\).
For \( C \) given by a parametrization \( \vec{r}(t) \), if \( \vec{r}'(t_0) \) is not \( \vec{0} \), then

Tangent Line to \( C \) at the point \( P = \vec{r}(t_0) \) is:

\[
L: \vec{x} = \vec{r}(t_0) + t \cdot \vec{r}'(t_0), \quad t \in \mathbb{R}
\]

Here \( \vec{r}'(t_0) = \vec{v}(t_0) \) is a direction vector of the tangent line.

A smooth parametrization on a curve \( C \) determines a direction on \( C \) from initial point to terminal point.

\( \vec{r}'(t) = \vec{v}(t) \): velocity vector on \( C \)

\( \vec{r}'(t) = \vec{T}(t) \): tangent vector to \( C \) at the point \( \vec{r}(t) \)

Closed Curve: \( \vec{r}(a) = \vec{r}(b) \) (initial point = terminal point)

simple closed curve

\( \infty \)

Closed but not simple

simple curve means there is no self-crossing.

clockwise oriented simple closed curve

counter-clockwise oriented simple closed curve.
Parametrizing Curves

1) Graph of a function $y = f(x)$ on $[a, b]$ 

$y = f(x)$ is parametrized as:

$C: \mathbb{R} \times \mathbb{R} = (x, y) = (x, f(x)), a \leq x \leq b$

Here $x$ is the parameter.

2) Circles

The circle $C$ is parametrized as:

$C: (x, y) = R(t) = (R \cos t, R \sin t), \quad 0 \leq t \leq 2\pi$

Direction on $C$ is counterclockwise.

$0 \leq t \leq \pi$ parametrizes upper semicircle

$\pi \leq t \leq 2\pi$ parametrizes lower semicircle

$0 \leq t \leq 4\pi$ parametrization traverses the circle twice.

$C: (x, y) = (a + R \cos t, b + R \sin t), \quad 0 \leq t \leq 2\pi$
3) **Ellipses**

Parametrization of $C$:

\[ (x, y) = \mathbf{r}(t) = (a \cos t, b \sin t), \quad 0 \leq t \leq 2\pi \]

\[ (x = a \cos t, y = b \sin t) \Rightarrow \frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{a^2 \cos^2 t}{a^2} + \frac{b^2 \sin^2 t}{b^2} = \cos^2 t + \sin^2 t = 1 \]

**Shifted Ellipse**:

\[ \frac{(x-c)^2}{a^2} + \frac{(y-d)^2}{b^2} = 1 \]

A parametrization is:

\[ C: \mathbf{r}(t) = (c + a \cos t, d + b \sin t), \quad \sigma \leq t \leq 2\pi \]
Parametrizing Curves

4) Lines, Rays, Line Segments (in $\mathbb{R}^2$ or $\mathbb{R}^3$)

For two points $A$ and $B$, the ray $[A\overrightarrow{AB}$
let $\overrightarrow{V} = \overrightarrow{AB}$

Parametrization of the line $L$: $P(t) = \overrightarrow{A} + t\overrightarrow{V}$, $t \in \mathbb{R}$
The ray $[AB]$:
the point $P(t) = \overrightarrow{A} + t\overrightarrow{V}$, $t \geq 0$
The line segment $[AB]$:
$P(t) = \overrightarrow{A} + t\overrightarrow{V}$, $0 \leq t \leq 1$

5) Reversing the direction of a parametrization:

For $C$: $\overrightarrow{P}(t) = (x(t), y(t), z(t))$, $a \leq t \leq b$
$\overrightarrow{P}_2(t) = \overrightarrow{P}(-t)$, $-b \leq t \leq -a$ is the parametrization of $C$ with reversed direction.

$-C = \overrightarrow{P}_2(t) = (x(-t), y(-t), z(-t))$, $-b \leq t \leq -a$

$\overrightarrow{P}(a)$
$\overrightarrow{P}(b)$

$-C$: same curve $C$ with reversed direction
Parametrizing Curves

6) Intersection curve of two surfaces

Example:
Parametrize the curve \( C \) which is the intersection of the two surfaces \( x + 2y + 3z = 6 \) and \( x^2 + (y - 1)^2 = 1 \) in \( \mathbb{R}^3 \).

Solution:
\[ x^2 + (y - 1)^2 = 1 \text{ is a cylinder.} \]
\[ x + 2y + 3z = 6 \text{ is a plane.} \]

\( C \) lies on the cylinder, so projection of \( C \) to \( xy \)-plane is the circle \( x^2 + (y - 1)^2 = 1 \).

**Parametrization of \( C_2 \):**
\[ (x, y) \in C_2 \]
\[ \Rightarrow (x, y) = \left( \cos t, \frac{1 + \sin t}{2} \right), \ 0 \leq t \leq 2\pi \]

\((x, y, z) \in C \Rightarrow (x, y) \in C_2\)
\[ \Rightarrow \ x = \cos t, \ y = 1 + \sin t, \ 0 \leq t \leq 2\pi \]

\[ x + 2y + 3z = 6 \Rightarrow z = \frac{6 - x - 2y}{3} = \frac{6 - \cos t - 2 - 2\sin t}{3} \]

Thus
\[ C = (x, y, z) = \left( \cos t, 1 + \sin t, \frac{6 - \cos t - 2 - 2\sin t}{3} \right), \ 0 \leq t \leq 2\pi \]
Example

Parametrize the intersection of the paraboloids $z = x^2 + 2y^2$ and $z = 6 - x^2 - y^2$.

Solution:

$(x, y, z) \in C \iff \begin{cases} 
\begin{align*}
2x^2 + 2y^2 &= 6 - x^2 - y^2 \\
\frac{x^2}{3} + \frac{y^2}{2} &= 1
\end{align*}
\end{cases}$

Eliminating $z$ using the 2 equations,

$x^2 + 2y^2 = 6 - x^2 - y^2$

$2x^2 + 3y^2 = 6$

$x^2 + y^2 = 3 + \sin^2 t$

$x = \sqrt{3} \cos t, \quad y = \sqrt{2} \sin t$

$C_2$ : projection of $C$ to $xy$-plane.

$C_2 = (x, y) = (\sqrt{3} \cos t, \sqrt{2} \sin t), \quad 0 \leq t \leq 2\pi$

Then $(x, y, z) \in C \Rightarrow (x, y) \in C_2 \Rightarrow x = \sqrt{3} \cos t, \quad y = \sqrt{2} \sin t$

$z = x^2 + 2y^2 \Rightarrow z = 3 \cos^2 t + 4 \sin^2 t = 3 + \sin^2 t$

$C = (x, y, z) = (\sqrt{3} \cos t, \sqrt{2} \sin t, 3 + \sin^2 t), \quad 0 \leq t \leq 2\pi$

Warning: Above projection of $C$ to $xy$-plane is all of $C_2$ as we can understand from the graph of the paraboloids.

If we eliminated $x$ from the 2 equations, we would get $C_3: \quad z = 3 + \frac{y^2}{2}$. Projection of $C$ to $yz$ plane is in $C_3$, but not all of $C_3$!!!