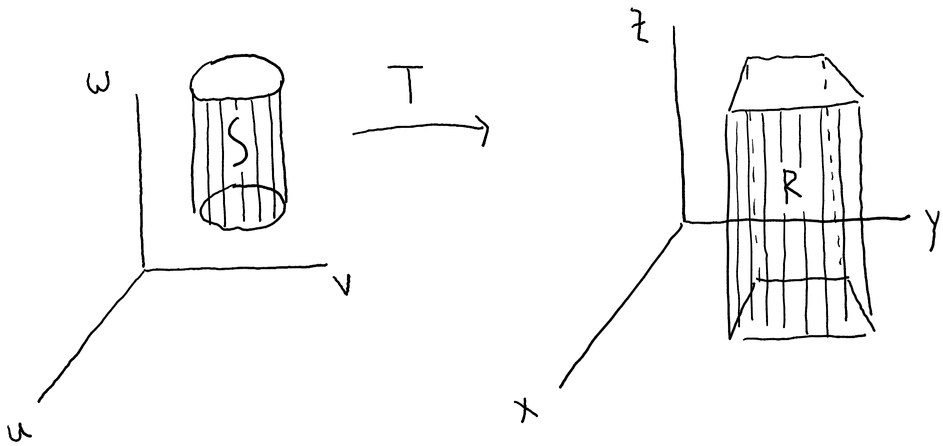
Lecture 19

Tuesday, 14 April 2020 00:17

Change of Variables in Triple Integrals



$x = g(u,v,w)$, $y = h(u,v,w)$ and $z = k(u,v,w)$

The Jacobian of T is

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

If T is a one-to-one C^1 transformation and f is continuous, then

$$\iiint_R f(x,y,z)\,dV = \iiint_S f(g(u,v,w), h(u,v,w), k(u,v,w)) \left| \frac{\partial(x,y,z)}{\partial(u,v,w)} \right| du\,dv\,dw$$

e.g. spherical coordinates

$x = \rho \sin\phi \cos\theta$, $y = \rho \sin\phi \sin\theta$, $z = \rho \cos\phi$

$$\frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)} = \begin{vmatrix} \sin\phi \cos\theta & -\rho \sin\phi \sin\theta & \rho \cos\phi \cos\theta \\ \sin\phi \sin\theta & \rho \sin\phi \cos\theta & \rho \cos\phi \sin\theta \\ \cos\phi & 0 & -\rho \sin\phi \end{vmatrix}$$

$$= (-1)^{3+1} \cos\phi \begin{vmatrix} -\rho\sin\phi\sin\theta & \rho\cos\phi\cos\theta \\ \rho\sin\phi\cos\theta & \rho\cos\phi\sin\theta \end{vmatrix} + (-1)^{3+3}(-\rho\sin\phi) \begin{vmatrix} \sin\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\sin\phi\cos\theta \end{vmatrix}$$

$$= \cos\phi \left(-\rho^2 \sin\phi \cos\phi\right) + (-\rho\sin\phi)\, \rho \sin\phi = -\rho^2 \sin\phi$$

$$\left| \frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)} \right| = \rho^2 \sin\phi$$

Ex: $\iiint_E x^2 y \, dV$ where E is the solid enclosed by the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

$x = au$, $y = bv$ and $z = cw$ $\qquad u^2 + v^2 + w^2 = 1$

$x = au$, $y = bv$ and $z = cw$

$u^2 + v^2 + w^2 = 1$

$$\left| \frac{\partial(x,y,z)}{\partial(u,v,w)} \right| = abc$$

$$\iiint_{u^2+v^2+w^2 \leq 1} (a^2u^2)(bv)\, abc \; du\, dv\, dw$$

$u = \rho \sin\phi \cos\theta$

$v = \rho \sin\phi \sin\theta$

$w = \rho \cos\phi$

$$= \int_0^{2\pi} \int_0^{\pi} \int_0^{1} (a^3 b^2 c)\, \rho^5 \underbrace{\sin^4\phi}_{\left(\frac{1-\cos 2\phi}{2}\right)^2} \cos^2\theta \sin\theta \; d\rho\, d\phi\, d\theta$$