Lecture 18

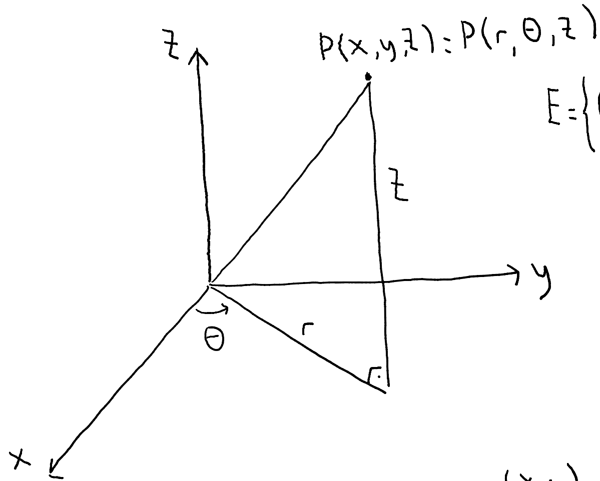
Monday, 13 April 2020 00:05

# Triple Integrals in Cylindrical and Spherical Coordinates

## Cylindrical Coordinates

$x = r\cos\theta \qquad z = z$

$y = r\sin\theta$

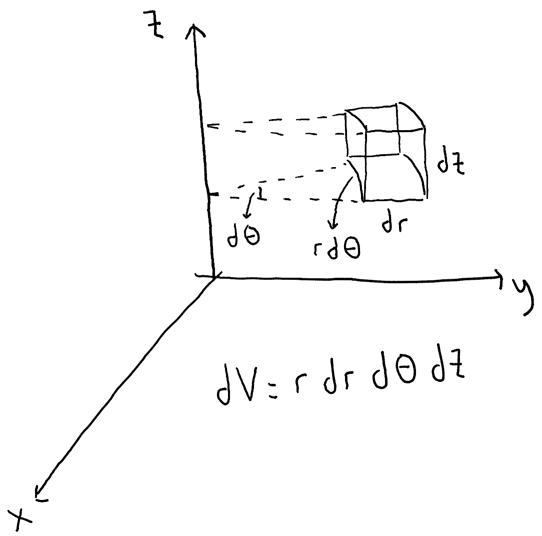



$$E = \{(x,y,z) \mid (x,y) \in D,\ u_1(x,y) \leq z \leq u_2(x,y)\}$$

$$D = \{(r,\theta) \mid \alpha \leq \theta \leq \beta,\ h_1(\theta) \leq r \leq h_2(\theta)\}$$

$$\iiint_E f(x,y,z)\,dV = \iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\right] dA$$

$$= \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r\cos\theta,\, r\sin\theta)}^{u_2(r\cos\theta,\, r\sin\theta)} f(r\cos\theta, r\sin\theta, z)\, r\,dz\,dr\,d\theta$$

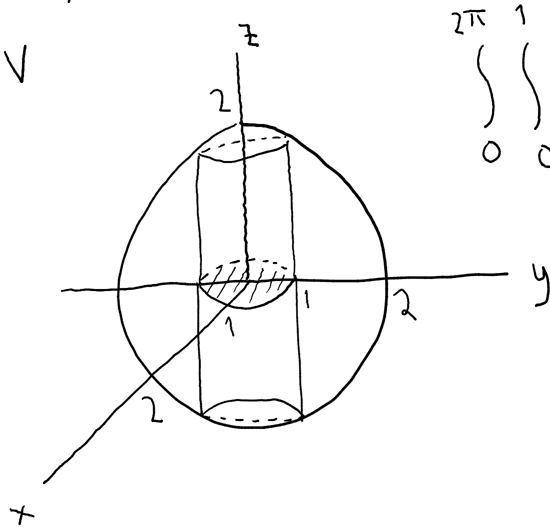



$dV = r\,dr\,d\theta\,dz$

Ex. Find the volume of the solid that lies within both

Ex: Find the volume of the solid that lies within both the cylinder $x^2+y^2=1$ and the sphere $x^2+y^2+z^2=4$.

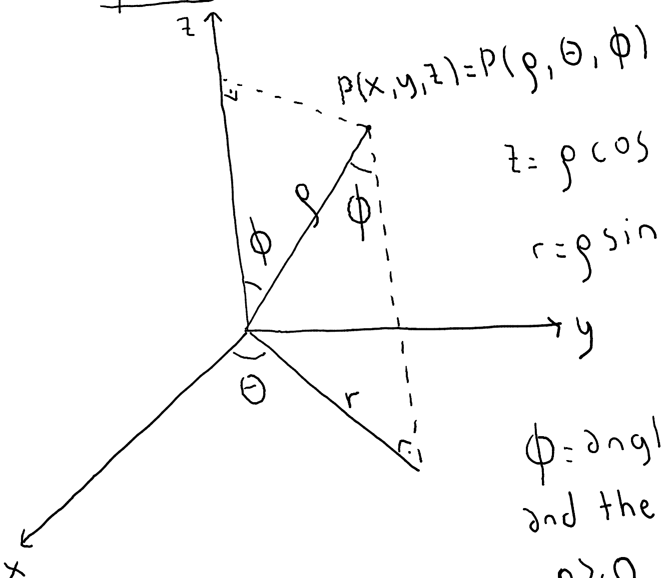
$$\iiint_E dV$$




$$\int_0^{2\pi}\int_0^1\int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} r\, dz\, dr\, d\theta$$

$$\underbrace{x^2+y^2+z^2=4}_{\downarrow} \qquad z=\pm\sqrt{4-(x^2+y^2)}$$

## Spherical Coordinates




$z=\rho\cos\phi$

$r=\rho\sin\phi$

$\rho$ = distance from P to the origin.

$\phi$ = angle between the positive z-axis and the line segment $\overrightarrow{OP}$

$\rho \geq 0 \qquad 0\leq\phi\leq\pi$

$$\left.\begin{aligned} x&=\rho\sin\phi\cos\theta\\ y&=\rho\sin\phi\sin\theta\\ z&=\rho\cos\phi\end{aligned}\right\} \Rightarrow \rho^2=x^2+y^2+z^2=r^2+z^2$$

$$r=\sqrt{x^2+y^2}=\rho\sin\phi$$

$$\tan\phi=\frac{r}{z}=\frac{\sqrt{x^2+y^2}}{z} \quad \text{and} \quad \tan\theta=\frac{y}{x}$$

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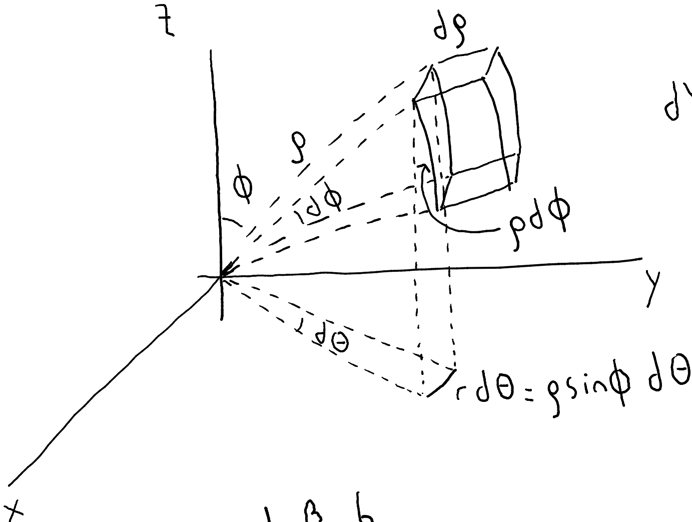
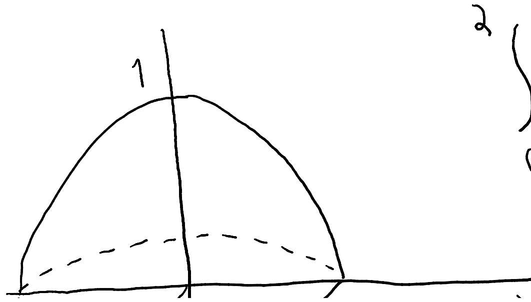



$$dV = \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$$

$$\iiint_E f(x,y,z)\,dV = \int_c^d \int_\alpha^\beta \int_a^b f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\,\rho^2\sin\phi \, d\rho \, d\theta \, d\phi$$

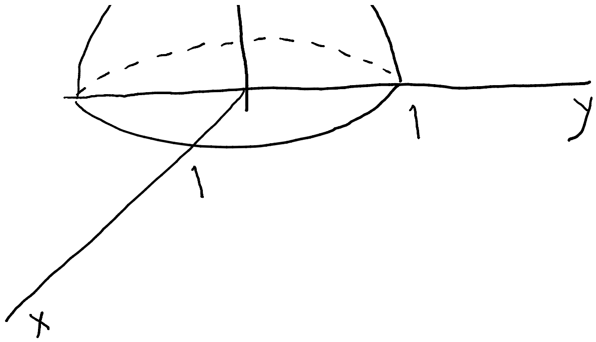
Usually, spherical coordinates are used in triple integrals when surfaces such as cones and spheres form the boundary of the region of integration.

Ex: $\iiint_H (x^2+y^2)\,dV$ where H is the hemispherical region that lies above the xy-plane and below the sphere $x^2+y^2+z^2=1$.




$$\int_0^{\frac{\pi}{2}} \int_0^{2\pi} \int_0^1 \rho^2 \sin^2\phi \, \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$$

$$\int^{\frac{\pi}{2}} \int^{2\pi} \int^1 \rho^3 \sin^3\phi \, d\rho \, d\theta \, d\phi$$




$$= \int_0^{\frac{\pi}{2}} \int_0 \int_0 \rho^3 \sin^3\phi \, d\rho \, d\theta \, d\phi$$

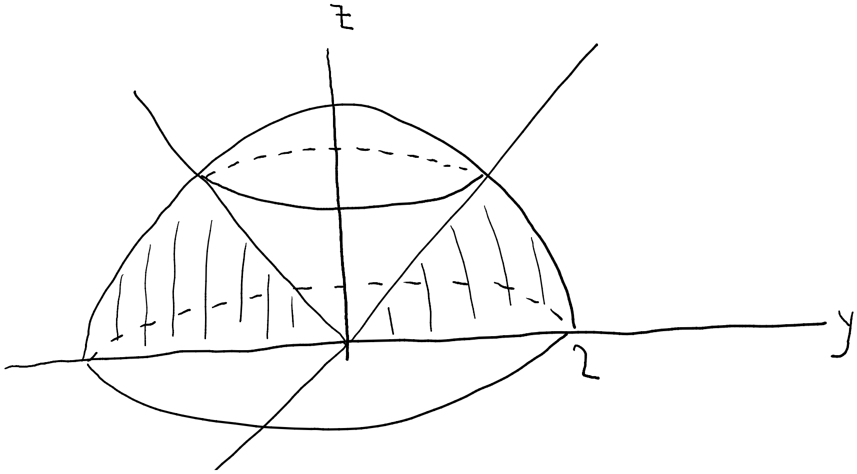
$$= \int_0^{\frac{\pi}{2}} \int_0^{2\pi} \frac{1}{4} \sin^3\phi \, d\theta \, d\phi$$

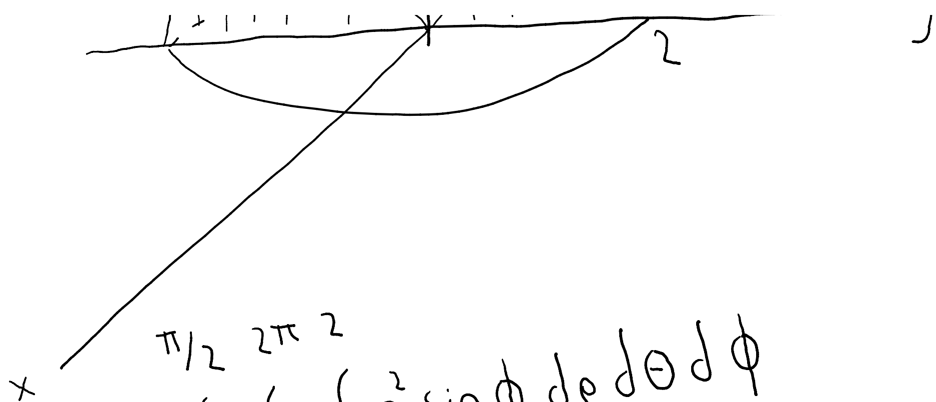
$$= \frac{2\pi}{4} \int_0^{\frac{\pi}{2}} \sin^3\phi \, d\phi$$

$$= \frac{\pi}{2} \int_0^{\frac{\pi}{2}} (1-\cos^2\phi) \sin\phi \, d\phi$$

$$(u = \cos\phi)$$

Ex: Find the volume of the solid that lies within the sphere $x^2+y^2+z^2=4$, above the $xy$-plane, and below the cone $z=\sqrt{x^2+y^2}$






$$\int_{\pi/4}^{\pi/2}\int_0^{2\pi}\int_0^2 \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$$

Ex: Find the volume of the region above the xy-plane, inside the cone $z = 2a - \sqrt{x^2+y^2}$ and inside the cylinder $x^2+y^2 = 2ay$

$$V = 2\int_0^{\pi/2} d\theta \int_0^{2a\sin\theta} \int_0^{2a-r} r \, dz \, dr.$$