Lecture 17

Sunday, 12 April 2020 01:01

Triple Integrals

If f is continuous on the rectangular box

$B = [a,b] \times [c,d] \times [r,s]$, then

$$\iiint_B f(x,y,z)\,dV = \int_r^s \int_c^d \int_a^b f(x,y,z)\,dx\,dy\,dz$$

$$= \int_a^b \int_r^s \int_c^d f(x,y,z)\,dy\,dz\,dx$$

$$= \int_c^d \int_a^b \int_r^s f(x,y,z)\,dz\,dx\,dy$$

We can iterate in any order we want.

All properties of double integrals have analogues for triple integrals:

A continuous function is integrable over a closed, bounded domain.

$$\iiint_D dV = \text{Volume of } D$$

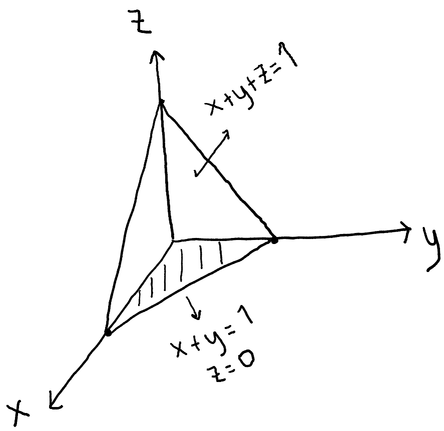
D

Triple integrals over more general domains can be defined similarly. For example, if

$E = \{ (x,y,z) \mid (x,y) \in P,\ u_1(x,y) \leq z \leq u_2(x,y) \}$ where P is the projection of E onto the xy-plane

$$\iiint_E f(x,y,z)\, dV = \iint_P \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\, dz \right] dA$$

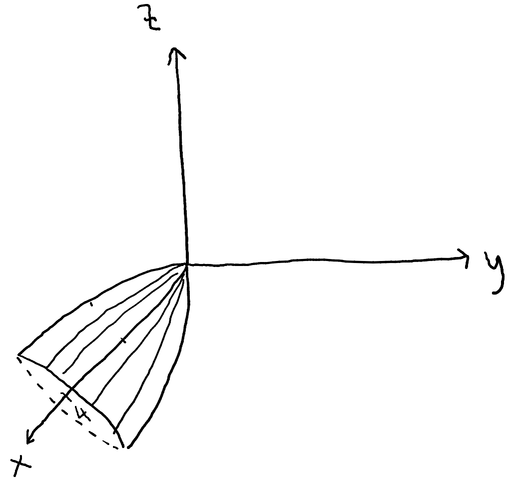
Ex: $\iiint_E z\, dV$ E is the solid tetrahedron bounded by the four planes $x=0, y=0, z=0$ and $x+y+z=1$.

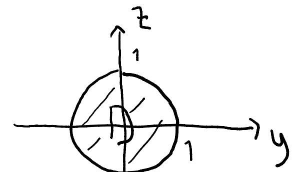


$$\int_0^1 \int_0^{1-x} \int_0^{1-x-y} z\, dz\, dy\, dx$$

Ex: $\iiint_E x\, dV$ E is bounded by the paraboloid $x = 4y^2 + 4z^2$ and the plane $x=4$.

t





$$\iint_{D}\int_{4y^2+4z^2}^{4} x\,dx\,dA = \frac{1}{2}\iint_{D}\left[4^2-(4y^2+4z^2)\right]dA$$

D: $4y^2+4z^2 \leq 4$

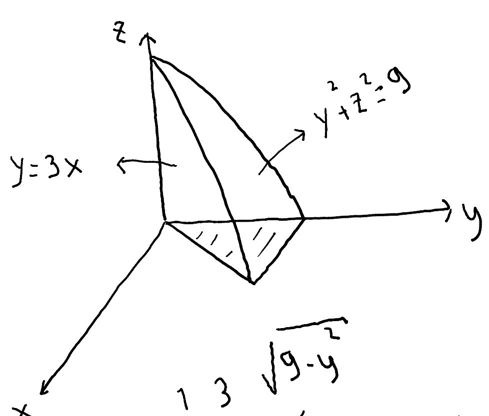
$y^2+z^2 \leq 1$

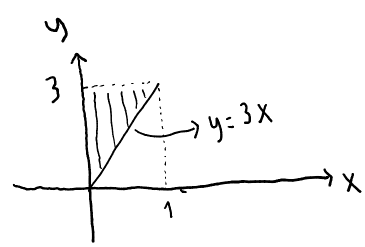
$y = r\cos\theta$

$z = r\sin\theta$

$$= 8\int_{0}^{2\pi}\int_{0}^{1}(1-r^4)\,r\,dr\,d\theta$$

Ex: $\iiint_{E} z\,dV$ E is bounded by the cylinder $y^2+z^2=9$ and the planes $x=0$, $y=3x$, and $z=0$ in the first octant.

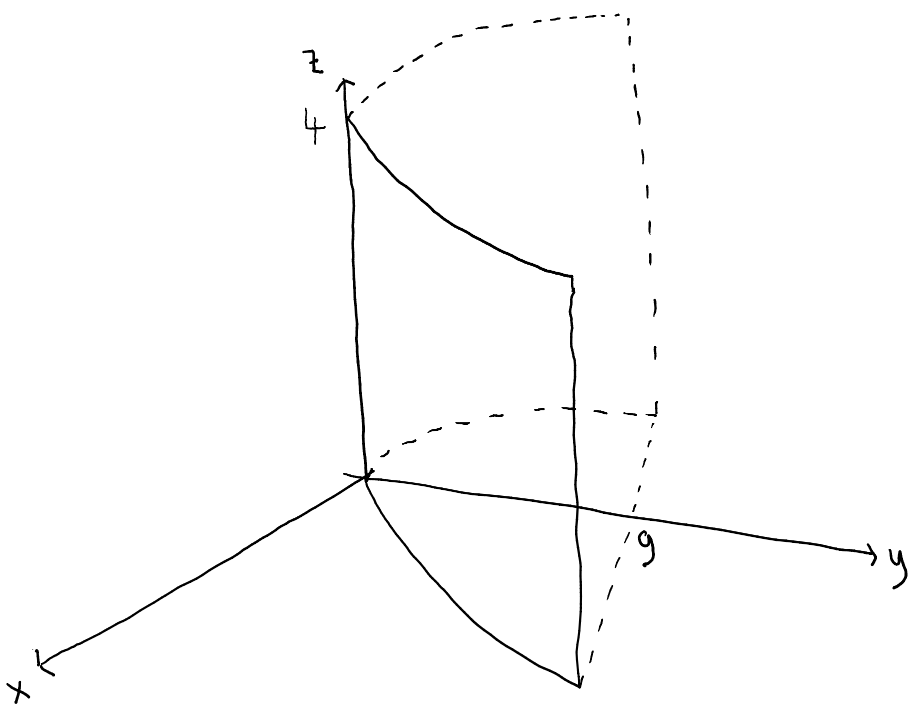


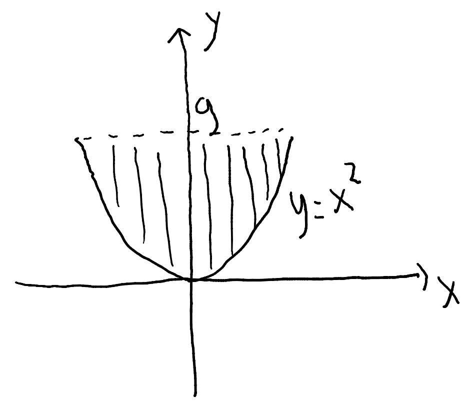


$$\int^{1}\int^{3}\int^{\sqrt{9-y^2}} z\,dz\,dy\,dx$$

$$\int_0^{x} \int_{3x} \int_0 z \, dz \, dy \, dx$$

Ex: Find the volume of the solid bounded by the cylinder $y = x^2$ and the planes $z = 0$, $z = 4$, and $y = 9$.





$$V = \iiint_E dV = \int_{-3}^{3} \int_{x^2}^{9} \int_0^{4} dz \, dy \, dx$$