Triple Integrals

If $f$ is continuous on the rectangular box $B = [a, b] \times [c, d] \times [r, s]$, then

$$\iiint_B f(x,y,z) \, dV = \int_a^b \int_c^d \int_r^s f(x,y,z) \, dx \, dy \, dz$$

We can iterate in any order we want.

All properties of double integrals have analogues for triple integrals:

A continuous function is integrable over a closed, bounded domain.

$$\iiint_D dV = \text{Volume of } D$$
Triple integrals over more general domains can be defined similarly. For example, if

\[ E = \{ (x,y,z) \mid (x,y) \in P, \quad u_1(x,y) \leq z \leq u_2(x,y) \} \]

where \( P \) is the projection of \( E \) onto the \( xy \)-plane,

\[ \iiint_E f(x,y,z) \, dV = \iiint_P \left[ \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z) \, dz \right] \, dA \]

Ex: \[ \iiint_E z \, dV \]

\( E \) is the solid tetrahedron bounded by the four planes \( x=0, y=0, z=0 \) and \( x+y+z=1 \).

\[ \iiint_E z \, dV = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} z \, dz \, dy \, dx \]

Ex: \[ \iiint_E x \, dV \]

\( E \) is bounded by the paraboloid \( z = 4y^2 + 4z^2 \) and the plane \( x=4 \).

\[ z \]
\[ \iiint_E x \, dx \, dy \, dz = \frac{1}{2} \iiint_D \left[ 4 - (4y^2 + 4z^2) \right] \, dA \]

\[ D : 4y^2 + 4z^2 < 4 \]
\[ y^2 + z^2 \leq 1 \]
\[ y = r \cos \theta \]
\[ z = r \sin \theta \]

\[ 2\pi \left( \int_0^1 \int_0^1 (1 - r^4) r^2 \, dr \right) \, d\theta \]

\[ = 8 \int_0^1 \int_0^1 (1 - r^4) r^2 \, dr \, d\theta \]

\[ = 0 \]

Ex: \[ \iiint_E z \, dV \]
\[ E \text{ is bounded by the cylinder } y^2 + z^2 = 9 \]
\[ \text{and the planes } x = 0, y = 3x, \text{ and } z = 0 \]
\[ \text{in the first octant.} \]
Ex: Find the volume of the solid bounded by the cylinder $y = x^2$ and the planes $z = 0$, $z = 4$, and $y = 9$.

$$V = \iiint_E dV = \iiint_0^{3x^2} 0^9 dxdzdy$$