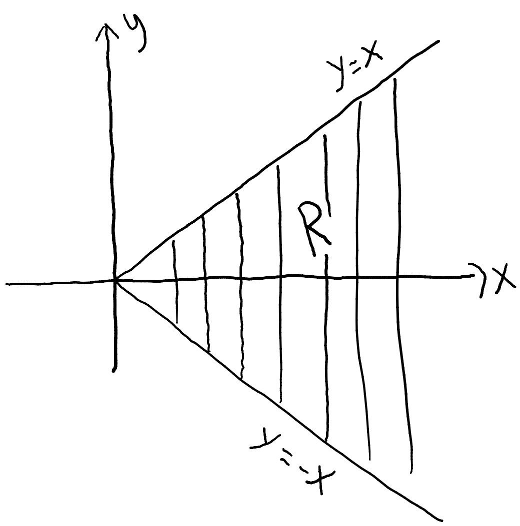
Lecture 14 Thursday, 9 April 2020 00:12

Improper Integrals and a Mean Value Theorem

Improper double integrals can arise if either the domain of integration is unbounded or the integrand is unbounded near any point of the domain or its boundary.

Ex: Evaluate $I = \iint_R e^{-x^2}\,dA$ R is the region where $x \geq 0$ and $-x \leq y \leq x$.



$$I = \int_0^\infty \int_{-x}^{x} e^{-x^2}\,dy\,dx = \int_0^\infty 2xe^{-x^2}\,dx = \lim_{t\to\infty} \int_0^t 2xe^{-x^2}\,dx$$

$$= \lim_{t\to\infty} -e^{-x^2}\Big|_0^t = \lim_{t\to\infty}(1-e^{-t^2}) = 1.$$

We cannot, in general, determine the convergence of the given integral by looking at the convergence of the given integral by looking at the convergence of iterations. The double integral may diverge even if its iterations converge (you can see such an integral on page 808 Exercise 21)

It can be shown that an improper double integral of $f(x,y)$ over D converges if the integral of $|f(x,y)|$ over D converges. Such double integrals are called absolutely convergent.

A Mean-Value Theorem for Double Integrals

If the function $f(x,y)$ is continuous on a closed, bounded, connected set D in the xy-plane, then there exists a point (x_0, y_0) in D such that

$$\iint_D f(x,y)\, dA = f(x_0,y_0) \cdot (\text{area of } D)$$

Definition: The average value or mean value of an integrable function $f(x,y)$ over the set D is the number

$$\bar{f} = \frac{1}{(\text{area of } D)} \iint_D f(x,y)\, dA.$$