Calculus of Functions of Several Variables

Recitation-08: Green's Theorem

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Last time: *Conservative Fields and Line Integrals of Vector Fields*

Topics to be covered:

15.4 Line Integrals of Vector Fields	**15.4: 4,6,8,9,13,22**
Ch. 16: Vector Calculus **16.3 Green's Theorem in the Plane**	**16.3: 1, 2, 3, 4, 5, 6, 7, 9**

Question 01

If C is the intersection of $z = \ln(1+x)$ and $y = x$ from $(0,0,0)$ to $(1,1,\ln 2)$ evaluate,

$$\int_C (2x\sin(\pi y) - e^z)dx + (\pi x^2 \cos \pi y - 3e^z)dy - xe^z dz.$$

soln

First check if the vector field $F = \langle \underbrace{2x\sin(y\pi) - e^z}_{F_1}, \underbrace{\pi x^2 \cos(\pi y) - 3e^z}_{F_2}, \underbrace{-xe^z}_{F_3} \rangle$ is conservative or not.

Notice that

- $\dfrac{\partial F_1}{\partial y} = 2x\pi\cos(\pi y) = \dfrac{\partial F_2}{\partial x} \quad \forall x,y \in Dom(F)$.
- $\dfrac{\partial F_1}{\partial z} = -e^z = \dfrac{\partial F_3}{\partial x} \quad \forall x,y$.
- $\dfrac{\partial F_3}{\partial y} = 0$ but $\dfrac{\partial F_2}{\partial z} = -3e^z \neq 0 \quad \leftarrow$ cond. fails !

So, F is not conservative !

Problem: It is still hard to use direct computation !

↳ New trick : modify F so that it becomes conservative !

- need an extra term in F_3 so that $\dfrac{\partial F_3}{\partial y} = -3e^z \leftarrow$ [Integrate wrt y gives $F_3(x,y,z) = -3ye^z + h(x,z)$]

hint !

Rewrite $\int_C F \cdot dr$ as follows:

$$\int_C F\cdot dr = \underbrace{\int_C (2x\sin(\pi y) - e^z)dx + (\pi x^2\cos(\pi y) - 3e^z)dy + (-xe^z - 3ye^z)dz}_{\mathrm{I}\ \checkmark} + \underbrace{\int_C 3ye^z\, dz}_{\mathrm{II}}$$

Evaluate I : Let $G = \langle 2x\sin(\pi y) - e^z, \pi x^2\cos(\pi y) - 3e^z, -xe^z - 3ye^z \rangle$.

Observe that, by defn of G, G satisfies all neccessary conds !

So, G may be conservative !

Let $G = \nabla \phi$. Then we have

$$\frac{\partial \phi}{\partial x} = G_1 = 2x\sin(\pi y) - e^z \qquad (1)$$

$$\frac{\partial \phi}{\partial y} = G_2 = \pi x^2 \cos(\pi y) - 3e^z \qquad (2)$$

$$\frac{\partial \phi}{\partial z} = G_3 = -xe^z - 3ye^z \qquad (3)$$

Integrate eqn (1) wrt x, we get

$$\boxed{\phi(x,y,z) = x^2\sin(\pi y) - xe^z + h(y,z)}$$

Then, by eqn (2),

$$\cancel{\pi x^2\cos(\pi y)} - 3e^z \underset{(2)}{=} \frac{\partial \phi}{\partial y} = \cancel{\pi x^2\cos(\pi y)} + \frac{\partial h}{\partial y} \qquad \forall x, y.$$

which gives $\frac{\partial h}{\partial y} = -3e^z$, and hence $\boxed{h(y,z) = -3ye^z + f(z)}$.

i.e., $\boxed{\phi(x,y,z) = x^2\sin(\pi y) - xe^z - 3ye^z + f(z)}$

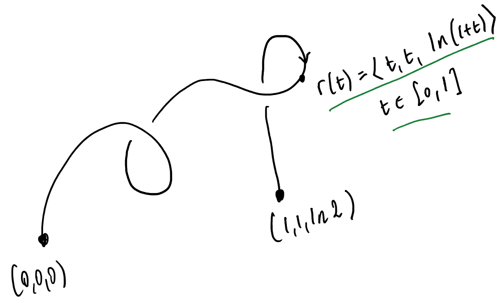
Now, using eqn (3), we get

$$-x\cancel{e^z} - 3y\cancel{e^z} \underset{(3)}{=} \frac{\partial \phi}{\partial z} = -\cancel{xe^z} - \cancel{3ye^z} + \frac{df}{dz}$$

$\Rightarrow \frac{df}{dz} = 0$, and so $f(z) = K$ for some constant K.

Just $K=0$; we have "a potential ϕ" for G: $\boxed{\phi = x^2\sin(\pi y) - xe^z - 3ye^z}$ s.t. $\nabla\phi = G$.

Therefore, by indep of path, we have



$$I = \int_C G \cdot dr = \phi(1,1,\ln 2) - \phi(0,0,0) = -8$$

Compute $\mathbb{I}$: Using the parametrization $x=t$, $y=t$, and $z=\ln(1+t)$,

$$\mathbb{I} = \int_C 3ye^z\,dz = \int_0^1 3t\,\cancel{(1+t)}\,\frac{1}{\cancel{(1+t)}}\,dt = 3/2$$

In total, we obtain:

$$\int_C F\cdot dr = \int_C G\cdot dr + \int_C 3ye^z dz = -8+3/2 = -\frac{13}{2}$$

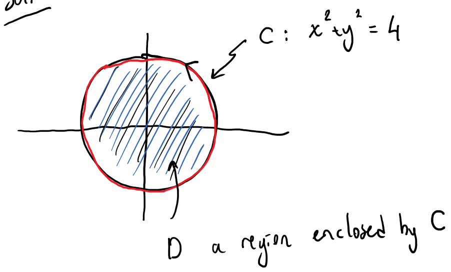
$\boxed{\text{Conservative part of } F}$

$\boxed{\langle 0,0,3ye^z\rangle\text{-part of } F}$

Question 02

Evaluate $\oint (3x + 4y)\,\mathrm{d}x + (2x + 3y^2)\,\mathrm{d}y$ around $x^2 + y^2 = 4$.

soln



Line integrals on a suitable closed curves $\xleftrightarrow{GT}$ Double integrals on some regions enclosed by those curves

Here, D is a reguler region with a boundry a curve $C : x^2+y^2=4$, which smooth, simple, and closed curve wrt c.c.w orientation (positive)

Then, by Green's thm, we have

$$\oint_C \underbrace{(3x+4y)}_{=P}\,dx + \underbrace{(2x+3y^2)}_{=Q}\,dy \overset{G.T.}{=} \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

$$= \iint_D (2-4)\,dA$$

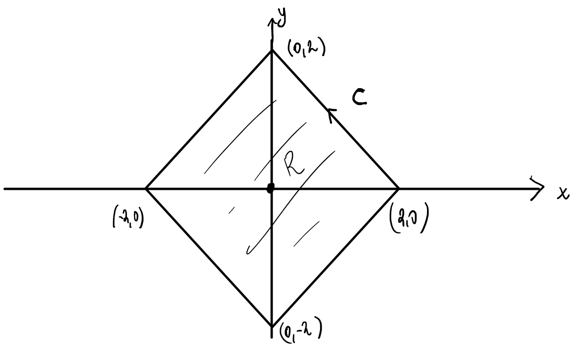
$$= -2 \underbrace{\iint_D 1 \cdot dA}_{=\text{area}(D)} = -8\pi \;//$$

Question 03

Given the vector field

$$\mathbf{F} = \frac{y\mathbf{i} - x\mathbf{j}}{x^2+y^2}, \leftarrow \left\langle \underbrace{\frac{y}{x^2+y^2}}_{F_1}, \underbrace{\frac{-x}{x^2+y^2}}_{F_2} \right\rangle$$

compute $\oint_C \mathbf{F} \bullet d\vec{r}$ where C is the curve enclosing the square with vertices $(2,0)$, $(0,2)$, $(-2,0)$, $(0,-2)$ oriented in the positive direction.



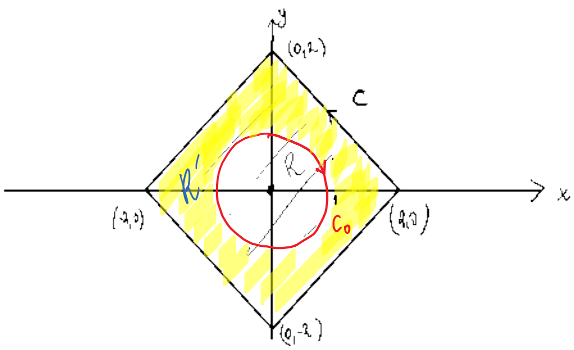
First observations:

- Let R be the region enclosed by C.
- R is <u>not</u> a regular region: F_1 and F_2 are not defined at the origin.

So, we can <u>not</u> use Green's thm

<u>Idea</u> Introduce a new region $R' \subset R$ such that

- It excludes the origin
- It is enclosed by a suitable boundary curve.



Let $C_0 : x^2+y^2 = 1$ be the unit circle inside R.

Define R' to be region enclosed by a curve

$$C' := C \cup C_0$$

which is pos-oriented, smooth, simple, and closed curve.

Observe that

now, a regular region

$$\frac{\partial F_2}{\partial x} = \frac{x^2-y^2}{(x^2+y^2)^2} = \frac{\partial F_1}{\partial y} \quad \forall x,y \in R'$$

Apply G.T. for the curve C':

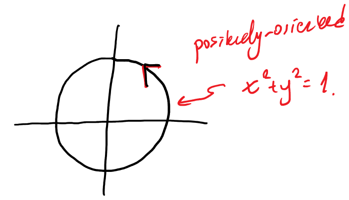
$$\oint_{\substack{C' \\ = \\ \partial R'}} F \cdot dr \overset{G.T}{=} \iint_{R'} \underbrace{\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)}_{=\ 0 \text{ on } R'} dA = 0$$

On the other hand,

$$0 = \oint_{C'} F \cdot dr = \oint_{C} F \cdot dr + \oint_{C_0} F \cdot dr \quad \text{where } C' = C \cup C_0$$

On the other hand, $\quad 0 = \oint_{C'} F\cdot dr = \oint_{C} F\cdot dr + \oint_{C_0} F\cdot dr \quad$ where $C' = C \cup C_0$

so, $(*)$ $\boxed{\oint_C F\cdot dr = -\oint_{C_0} F\cdot dr = \oint_{-C_0} F\cdot dr}$ where $-C_0$:



so, It is enough to compute $\oint_{-C_0} F\cdot dr$: Use direct computation with the parametrization

$x = \cos t,\ y = \sin t,\quad t\in[0,2\pi]$.

$$\Rightarrow \oint_{-C_0} F\cdot dr = \int_{-C_0} \frac{y}{x^2+y^2}\,dx - \frac{x}{x^2+y^2}\,dy = \int_0^{2\pi} \frac{\sin t\,(-\sin t\,dt)}{1} - \frac{\cos t\,(\cos t\,dt)}{1}$$

$$= -\int_0^{2\pi} dt$$

$$= -2\pi$$

Due to $(*)$, we have

$$\oint_C F\cdot dr = -2\pi$$

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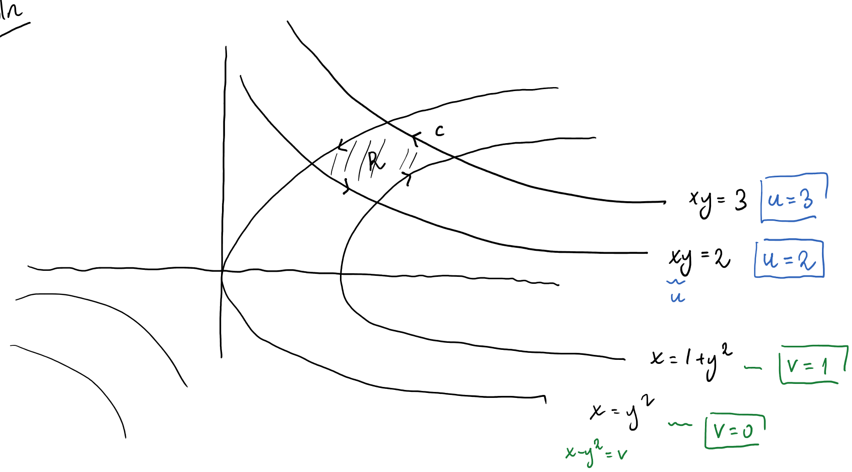
Question 04

Let C be the boundary of the region bounded by $x=y^2$, $x=1+y^2$, $xy=2$, $xy=3$ which is oriented counterclockwise. Evaluate $\oint_C (e^{x^3}-\frac{2}{3}y^3)\,dx+(\sin y^3+\frac{1}{2}x^2)\,dy$.

soln



- C is pw-smooth, simple, closed curve.
- C bounds a region R on which the v.f.

$$\vec{F}=\langle \underbrace{e^{x^3}-\frac{2}{3}y^3}_{P},\ \underbrace{\sin y^3+\frac{1}{2}x^2}_{Q}\rangle$$

is smooth.

By Green's thm, we have

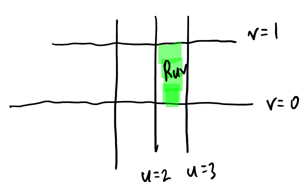
$$\oint_C P\,dx+Q\,dy \overset{G.T.}{=} \iint_R \frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\,dA = \underbrace{\iint_R x+2y\ \ dA}_{\text{Use change of coordinates}}$$

Let $xy=u$ and $x-y^2=v$. Then we get

(1) $\dfrac{\partial(u,v)}{\partial(x,y)}=\det\begin{bmatrix} y & x \\ 1 & -2y\end{bmatrix}=-2y^2-x$, and hence the area element is

$$dA=\left|\frac{\partial(x,y)}{\partial(u,v)}\right|du\,dv=\frac{1}{|x+2y^2|}du\,du=\frac{1}{x+2y^2}du\,dv$$

(2) R becomes



so, compute

$$\iint_R x+2y^2\ dA=\iint_{R_{uv}} \cancel{x+2y^2}\,\frac{du\,dv}{\cancel{x+2y^2}}=\iint_{R_{uv}} 1.\,du\,dv=\text{area}(R_{uv})=1$$

Question 05

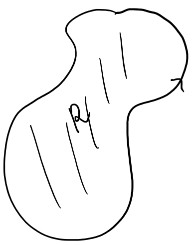
Find the area enclosed by the curve

$$\vec{r}(t) = \left\langle \frac{2\cos t - \sin t}{2},\ \sin t \right\rangle, \quad t \in [0, 2\pi].$$

Hint: Apply Green's theorem for the vector field $\mathbf{F} = \frac{1}{2}(-y\vec{i} + x\vec{j})$ along the curve $\vec{r}(t)$.

Soln

Recall: $\displaystyle\oint_C F\cdot dr = \frac{1}{2}\oint_C \underbrace{-y}_{P}\,dx + \underbrace{x}_{Q}\,dy \overset{G.T}{=} \frac{1}{2}\iint_R 1-(-1)\,dA = \iint_R 1.\,dA = Area(R)$



C given by $\vec{r}(t)$

So,

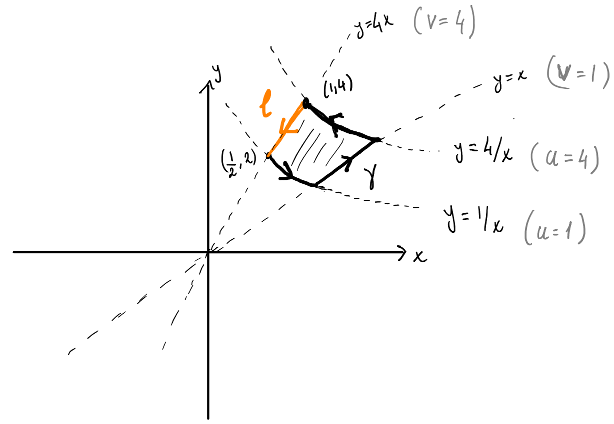
$$Area(R) = \frac{1}{2}\oint_C -y\,dx + x\,dy \quad \text{with} \quad x = \frac{2\cos t - \sin t}{2},\ y = \sin(t)$$

$$= \frac{1}{2}\int_0^{2\pi} -\sin(t)\left(\frac{-2\sin t - \cos t}{2}\right)dt + \frac{2\cos t - \sin t}{2}\cos t\,dt$$

$$= \frac{1}{4}\int_0^{2\pi} \underbrace{2\sin^2 t + 2\cos^2 t}_{=2}\,dt$$

$$= \pi.$$

Question 06

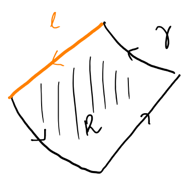


Evaluate $\displaystyle\int_{(\frac{1}{2},2)}^{(1,4)} -\frac{y^2}{x}\,dx + y\ln x\,dy$ along the curve γ where

sln Do not have a closed curve, but γ can be made closed

Introduce a line segment ℓ joining $(\frac{1}{2},2)$ and $(1,4)$, we get

C:



Let $C := \gamma \cup \ell$. Then

C is a closed, smooth, simple, pos-oriented curve.

Observe that

$$(*) \quad \underbrace{\int_C F\cdot dr}_{\text{I}} = \underbrace{\int_\gamma F\cdot dr} + \underbrace{\int_\ell F\cdot dr}_{\text{II}}$$

$\boxed{\text{Compute } \oint_C F\cdot dr}$

$\boxed{\text{Compute } \int_\ell F\cdot dr}$

where $F = \left\langle -\dfrac{y^2}{x}, \; y\ln x \right\rangle$, $F_1 = -\dfrac{y^2}{x}$, $F_2 = y\ln x$

Compute I : Using Green's thm, we have

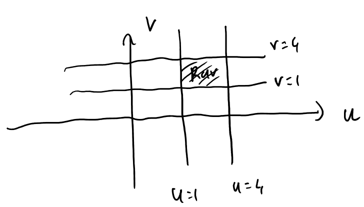
$$\oint_C F\cdot dr = \iint_R \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\, dA = 3\iint_R \frac{y}{x}\, dA$$

Let $xy = u$ and $\dfrac{y}{x} = v$. Then

(i) $$\frac{\partial(u,v)}{\partial(x,y)} = \det\begin{bmatrix} y & x \\ -\frac{y}{x^2} & \frac{1}{x}\end{bmatrix} = 2\frac{y}{x}, \text{ and hence}$$

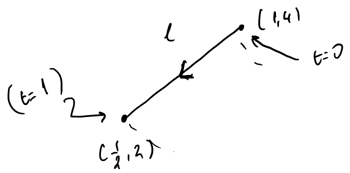
$$dA = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv = \frac{y}{2x}\, du\,dv, \quad (x,y\geq 0)$$

(ii) Region becomes



Then $$3\iint_R \frac{y}{x}\,dA = 3\iint_{R_{uv}} \frac{y}{\cancel{x}}\,\frac{\cancel{y}}{2x}\,du\,dv = \frac{3}{2}\iint_{R_{uv}} dA_{u,v} = \frac{3}{2}\,\text{area}(R_{uv}) = 27/2$$

Compute II : Parametrize ℓ as follows: $\langle x, y\rangle = \langle 1,4\rangle + t\left\langle -\frac{1}{2}, -2\right\rangle$, $t\in[0,1]$



i.e. $x = 1-\frac{t}{2}$, $y = 4-2t$

$\left[\, dx = -\frac{dt}{2}, \quad dy = -2\,dt \,\right]$

Then we obtain

$$\int_\ell F\cdot dr \underset{\text{exercise}}{=} \dots = 9 - 2\ln(2)$$

Totally, we get, from (*),

$$\int_\gamma F\cdot dr = \oint_C F\cdot dr - \int_\ell F\cdot dr \qquad \text{where } C = \gamma\cup\ell.$$

$$= \frac{27}{2} - (9 - 2\ln(2))$$

$$= \frac{9}{2} + 2\ln(2)$$