Calculus of Functions of Several Variables

Recitation-07: Conservative Fields and Line Integrals of Vector Fields

Kadri İlker Berktav
berktav@metu.edu.tr

Course webpage: http://ma120.math.metu.edu.tr/

Department of Mathematics: http://math.metu.edu.tr/

For more Online course materials:
http://math.metu.edu.tr/en/service-courses

METU | Mathematics Department

Last time: Vector functions and curves, line integrals

Topics to be covered:

16.1 Gradient, Divergence, and Curl	**16.1: 3,4**
15.2 Conservative Fields	**15.2: 2,6,9**
15.4 Line Integrals of Vector Fields	**15.4: 4,6,8,9,13,22**

Question 01

Determine whether the given vector is conservative find a potential if it is.

✓ (a) $\mathbf{F}(x,y,z) = e^{x^2+y^2+z^2}(xz\mathbf{i} + yz\mathbf{j} + xy\mathbf{k})$

(b) $\mathbf{F}(x,y) = (y\cos x - \cos y)\mathbf{i} + (\sin x + x\sin y)\mathbf{j}$

soln

(a) Recall: "the necessary conds." for a v.f. $F = \langle F_1, F_2, F_3 \rangle$ to be conservative on $Dom(F)$:

(i) $\left.\frac{\partial F_1}{\partial y}\right|_p = \left.\frac{\partial F_2}{\partial x}\right|_p$, (ii) $\left.\frac{\partial F_1}{\partial z}\right|_p = \left.\frac{\partial F_3}{\partial x}\right|_p$ and (iii) $\left.\frac{\partial F_2}{\partial z}\right|_p = \left.\frac{\partial F_3}{\partial y}\right|_p$ for all points $p=(x,y,z) \in Dom(F)$)

Rmk: . If one of them fails, "game over"

. If they hold $\xrightarrow{\text{possible conclusions}}$

(a) still not sufficient to %100 ensure that F is consv. on $Dom(F)$

(b) We just say that F may be conservative on $Dom(F)$

(c) On "nice" domains, Satisfying those conds. on D will be enough to ensure that F is conservative on $Dom(F)$.

In our case, $F_1(x,y,z) = e^{x^2+y^2+z^2}xz$, $F_2(x,y,z) = e^{x^2+y^2+z^2}yz$ and $F_3(x,y,z) = e^{x^2+y^2+z^2}xy$. with $Dom(F) = \mathbb{R}^3$.

Observe that

(i) $\left.\begin{aligned} \frac{\partial F_1}{\partial y} &= e^{x^2+y^2+z^2} \cdot 2y\,xz \\ \frac{\partial F_2}{\partial x} &= e^{x^2+y^2+z^2} \cdot 2x\,yz \end{aligned}\right\}$ $\left.\frac{\partial F_1}{\partial y}\right|_{p=(x,y,z)} = \left.\frac{\partial F_2}{\partial x}\right|_{p=(x,y,z)}$ with $p \in Dom(F) = \mathbb{R}^3$.

(ii) $\left.\begin{aligned} \frac{\partial F_1}{\partial z} &= x\,e^{x^2+y^2+z^2} + e^{x^2+y^2+z^2} \cdot 2z\,xz \\ \frac{\partial F_3}{\partial x} &= y\,e^{x^2+y^2+z^2} + e^{x^2+y^2+z^2}\,2x^2y \end{aligned}\right\}$

At $p_0 = (0,1,0) \in \mathbb{R}^3$, we have

$\left.\frac{\partial F_1}{\partial z}\right|_{p_0} = 0$, but $\left.\frac{\partial F_3}{\partial x}\right|_{p_0} = e$

$\Rightarrow \left.\frac{\partial F_1}{\partial z}\right|_{p_0} \neq \left.\frac{\partial F_3}{\partial x}\right|_{p_0}$

i.e. one of the nec. conds fails !

so, F is not conservative on $Dom(F)$.

(b) Recall that the necessary cond for a conservative v.f. $F = \langle F_1, F_2 \rangle$ are

$$\left.\frac{\partial F_1}{\partial y}\right|_p = \left.\frac{\partial F_2}{\partial x}\right|_p \quad \text{for all } p \in Dom(F).$$

In our case, $F = \langle \underbrace{y\cos x - \cos y}_{F_1(x,y)}, \underbrace{\sin x + x\sin y}_{F_2(x,y)} \rangle$ is given. (here $Dom(F) = \mathbb{R}^2$)

Check that

$$\frac{\partial F_1}{\partial y} = \cos x + \sin x = \frac{\partial F_2}{\partial x} \quad \text{for all } (x,y) \in \mathbb{R}^2.$$

which means that F can be conservative.

Assume $\boxed{F = \nabla \phi}$ for some smooth func. $\phi(x,y)$. (called "a potential func" for F on $Dom(F)$)

To find ϕ:

$$F = \langle F_1, F_2 \rangle = \left\langle \frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y} \right\rangle \iff \begin{array}{ll} (i) & \dfrac{\partial \phi}{\partial x} = y\cos x - \cos y \\ (ii) & \dfrac{\partial \phi}{\partial y} = \sin x + x\sin y \end{array}$$

Start with integrating eqn (i) wrt x, then we get

$$\boxed{\phi(x,y) = y\sin x - x\cos y + C(y)} \quad \text{for some func. } c(y)$$

"general form"

Then using eqn (ii), we have

$$\cancel{\sin x} + \cancel{x\sin y} = \frac{\partial \phi}{\partial y} = \cancel{\sin x} + \cancel{x\sin y} + \frac{dc}{dy}$$

which gives $\dfrac{dc}{dy} = 0$

So, $c(y) = K$ for some constant K.

Since we are looking for a potential func., just take $K=0$ and hence

$$\boxed{\phi(x,y) = y\sin x - x\cos y} \quad \text{s.t.} \quad \nabla\phi = F$$

So, F is conservative on $Dom(F) = \mathbb{R}^2$.

$\square$

Question 02

(a) Find a potential function for the vector field $\mathbf{F}(x, y) = 2xy\mathbf{i} + (x^2 + y^2)\mathbf{j}$ $= \langle F_1, F_2 \rangle$

(b) For the vector field in part (a), compute the line integral on C, where C is the portion of the curve $\sqrt{x} + xy + \sqrt{y} = 7$ starting at $(4, 1)$ and ending at $(1, 4)$.

(a) First, check the necessard cond for F to be conservative:

here, $F_1(x,y) = 2xy$ and $F_2(x,y) = x^2+y^2$ where $Dom(F) = \mathbb{R}^2$.

Observe that $\left.\frac{\partial F_2}{\partial x}\right|_{p=(x,y)} = 2x = \left.\frac{\partial F_1}{\partial y}\right|_{p=(x,y)}$ for all $p=(x,y) \in \mathbb{R}^2$.

so, F can be conservative on $Dom(F)$

Assume $F = \nabla\phi$ for some potential func. ϕ. Then we have

$$F = \nabla\phi \iff \begin{aligned} F_1 &= 2xy = \frac{\partial\phi}{\partial x} \quad (i) \\ F_2 &= x^2+y^2 = \frac{\partial\phi}{\partial y} \quad (ii) \end{aligned}$$

Integrate both sides of (i) wrt x gives $\boxed{\phi(x,y) = x^2y + h(y)}$ for some func. $h(y)$

some arbitrary funct to be determined.

To determine $h(y)$, use eqn (ii):

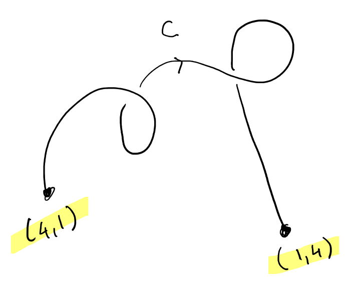
$$x^2+y^2 \underset{(ii)}{=} \frac{\partial\phi}{\partial y} = x^2 + \frac{dh}{dy} \implies \frac{dh}{dy} = y^2 \text{ and so,}$$

for all $(x,y) \in Dom(F)$ $\qquad \boxed{h(y) = \frac{y^3}{3} + C}$ for some constant C.

Just take $C = 0$, then we get a potential funct. for F:

$$\boxed{\phi(x,y) = x^2y + \frac{y^3}{3}}$$ where $\nabla\phi = F$ (so, F is conservative everywhere)

(b)



want to compute $\int_C \vec{F} \cdot d\vec{r}$

Observe that

(i) this part of C is a smooth curve.

$C: \sqrt{x} + xy + \sqrt{y} = 7$

Observe that

(i) this part of C is a smooth curve.

(ii) F is conservative.

So, by "independence of path", we have

$$\int_C \vec{F} \cdot d\vec{r} = \phi(4,1) - \phi(1,4) \qquad \text{where } \phi(x,y) = x^2 y + y^3/3$$

$$= 4 + \frac{64}{3} - 16 - \frac{1}{3}$$

$$= 9$$

Question 03

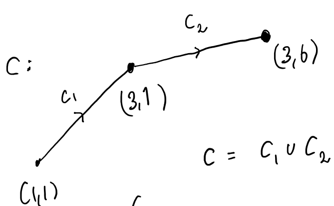
Evaluate $\int_C \frac{1}{xy}dx + \frac{1}{x+y}dy$ along the path from $(1,1)$ to $(3,1)$ to $(3,6)$ using straight line segments.

soln Here, we have a v.f. $\vec{F} = \left\langle \underbrace{\frac{1}{xy}}_{F_1(x,y)} , \underbrace{\frac{1}{x+y}}_{F_2(x,y)} \right\rangle$

(i) Check the necessary cond. of F :

$$\left.\begin{array}{l} \dfrac{\partial F_2}{\partial x} = -\dfrac{1}{(x+y)^2} \quad \text{and} \quad \dfrac{\partial F_1}{\partial y} = -\dfrac{1}{xy^2} \\ \text{e.g. at } p=(1,1), \quad \left.\dfrac{\partial F_2}{\partial x}\right|_p = -\dfrac{1}{2} \quad \text{but} \quad \dfrac{\partial F_1}{\partial y} = -1 \end{array}\right\} \Rightarrow F \text{ is } \underline{\underline{\text{not}}} \text{ conservative.}$$

(ii) Use direct computation:



$C = C_1 \cup C_2$.

$$\left(\text{so, } \int_C F\cdot dr = \int_{C_1} F\cdot dr + \int_{C_2} F\cdot dr \right)$$

- Need to parametrize line segments :
- C_1 : $\langle x,y \rangle = \langle 1,1 \rangle + t\langle 2,0 \rangle$, $t \in [0,1]$

here, $x = 1+2t$ and $y = 1$

so, $dx = 2dt$ and $dy = 0$.

- C_2 : $\langle x,y \rangle = \langle 3,1 \rangle + t\langle 0,5 \rangle$, $t \in [0,1]$

where $x = 3$ and $y = 1+5t$

so, $dx = 0$ and $dy = 5dt$

Then we get

$$\int_{C_1} F\cdot dr = \int_{C_1} F_1\,dx + F_2\,dy = \int_{C_1} \frac{1}{xy}dx + \frac{1}{x+y}dy = \int_0^1 \frac{1}{(1+2t)} 2dt + 0 = \ln(3)$$

$$\int_{C_2} F\cdot dr = \int_{C_2} \frac{1}{xy}dx + \frac{1}{x+y}dy = \int_0^1 0 + \frac{1}{4+5t} 5dt = \ln(9) - \ln(4)$$

so, $$\int_{C=C_1\cup C_2} F\cdot dr = \ln(3) + \ln(9) - \ln(4)$$

$\square$

Question 04

The field $\mathbf{F}(x, y, z) = (axy + z)\mathbf{i} + x^2\mathbf{j} + (bx + 2z)\mathbf{k})$ is conservative.

(a) Find a and b and a potential for $\mathbf{F}$.

(b) Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is the curve from $(1, 1, 0)$to $(0, 0, 3)$ that lies on the intersection of the surfaces $2x + y + z = 3$ and $9x^2 + 9y^2 + 2z^2 = 18$, in the first octant.

(a) Since $F \overset{= \langle F_1, F_2, F_3 \rangle}{}$ is conservative, $F = \nabla \phi$ for some func. ϕ and also

$$(i) \ \frac{\partial F_1}{\partial y}\Big|_p = \frac{\partial F_2}{\partial x}\Big|_p , \quad (ii) \ \frac{\partial F_2}{\partial z}\Big|_p = \frac{\partial F_3}{\partial y}\Big|_p \quad \text{and} \quad (iii) \ \frac{\partial F_1}{\partial z}\Big|_p = \frac{\partial F_3}{\partial x}\Big|_p$$

for all $p \in \mathrm{Dom}(\bar{F})$. , $F_1(x,y,z) = axy + z$, $F_2(x,y,z) = x^2$, $F_3(x,y,z) = bx + 2z$.

From (i), $\quad ax = 2x \quad \forall x \implies \boxed{a = 2}$ (or $x = 0$)

From (ii), $\quad 0 = 0$

From (iii), $\quad \boxed{1 = b}$

so, $$\bar{F} = \langle 2xy + z, \ x^2, \ x + 2z \rangle = \nabla \phi = \left\langle \frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z} \right\rangle$$

which means

$$(1) \ \frac{\partial \phi}{\partial x} = 2xy + z \qquad (2) \ \frac{\partial \phi}{\partial y} = x^2 \quad \text{and} \quad (3) \ \frac{\partial \phi}{\partial z} = x + 2z .$$

Need to find such ϕ :

Start with integrating eqn. (2) wrt y, then we get

$$\phi(x,y,z) = yx^2 + h(x,z) \quad \text{for some func. } h(x,z).$$

Using eqn (1), we have

$$\cancel{2xy} + z \underset{(1)}{=} \frac{\partial \phi}{\partial x} = \cancel{2xy} + \frac{\partial h}{\partial x} , \quad \text{and so} \quad \frac{\partial h}{\partial x} = z . \quad (*)$$

Integrating (*) wrt x gives

$$h(x,z) = xz + \underbrace{c(z)}_{} \quad \text{for some func. } c(z)$$

so far we get

$$\boxed{\phi(x,y,z) = yx^2 + xz + c(z)} \quad \text{"general form"}$$

To determine $c(z)$, we use eqn (3):

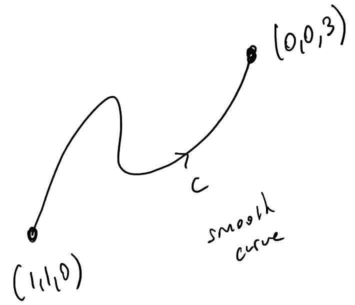
$$\cancel{x} + 2z \underset{(3)}{=} \frac{\partial \phi}{\partial z} = \cancel{x} + \frac{dc}{dz} \qquad \forall\, p = (x,y,z) \in \mathbb{R}^3$$

which gives $\frac{dc}{dz} = 2z$, and so $\boxed{c(z) = z^2 + K}$ for some constant K.

Just take $K=0$, then we get a potential func. for F:

$$\boxed{\phi(x,y,z) = yx^2 + xz + z^2} \qquad \text{where } \nabla\phi = F.$$

(b)



Since F is conservative, by using independence of path, we have

$$\int_C \vec{F} \cdot dr = \phi(0,0,3) - \phi(1,1,0) = 9 - 1 = 8$$

$\square$

Exercises

1. Consider the vector field

$$\mathbf{F} = \left(2xz^3 + 2xy\cos(x^2y)\right)\mathbf{i} + \left(\frac{z}{1+y^2} + x^2\cos(x^2y)\right)\mathbf{j} + \left(\arctan y + 1 + 3x^2z^2\right)\mathbf{k}$$

   (a) Find a potential function $f(x, y, z)$ for $\mathbf{F}$.

   (b) Compute $\int_C \mathbf{F} \bullet d\vec{r}$ where C is the curve given by

$$x = \sqrt{\pi}t^{1/3} + \sin(t - t^2),\ y = t^2,\ z = \sin(\frac{\pi}{2}t),\quad 0 \le t \le 1.$$

   (c) Let C' be the curve of intersection of the ellipsoid $2x^2 + 5y^2 + 3z^2 = 1$ and the plane $y + 2x = 0$. Compute $\oint_{C'} \mathbf{F} \bullet d\vec{r}$ for either orientation on C'. *(Hint: Recall the path-independence in case of closed curves, then the answer will be immediate.)*

2. $\mathbf{F}(x, y, z) = \dfrac{1}{(x^2 + y^2 + z^2)^{3/2}}(-x\mathbf{i} - y\mathbf{j} - z\mathbf{k})$ along the curve $\mathbf{r}(\mathbf{t}) = (1+t)\mathbf{i} + t^3\mathbf{j} + t\cos\pi t\mathbf{k}$, $\quad 0 \le t \le 1$.

   Find the work done by the force on the object. *(Hint: Evaluate* $\int_C \mathbf{F}\cdot d\mathbf{r}$ *by using the path independence for the curve C with the given parametrization* $\mathbf{r}(\mathbf{t})$*.)*