Calculus of Functions of Several Variables
Recitation-06: Vector functions and curves, line integrals

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Topics to be covered:

Ch. 11: Vector Functions and Curves	
11.1 Vector Functions of One Variable	**11.1: 8,10,16,18**
11.3 Curves and Parametrizations	**11.3: 1,2,3,4,6,8,17,18,24**
15.3 Line Integrals	**15.3: 2,6,8,13,14**

Question 01

Parametrize the curve of intersection of the given surfaces.

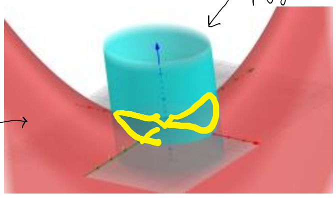
(a) The elliptical cylinder $\frac{x^2}{9} + \frac{y^2}{4} = 1$ with the surface $z = xy$.

(b) The cone $x^2 + y^2 = z^2$ with $\frac{x}{y} = \tan z$.

(c) The paraboloid $z = 4x^2 + y^2$ and the parabolic cylinder $y = x^2$.

$y = x^2$

ellip. cyl.



$z = xy$

(a) Let $x = 3\cos t$, $y = 2\sin t$ with $0 \le t \le 2\pi$

Then $\frac{x^2}{9} + \frac{y^2}{4} = 1 \iff \cos^2 t + \sin^2 t = 1$

So, $z = xy \iff z = 6\cos t \cdot \sin t$

The curve C can be parametrized as

$C(t) = (x(t), y(t), z(t))$

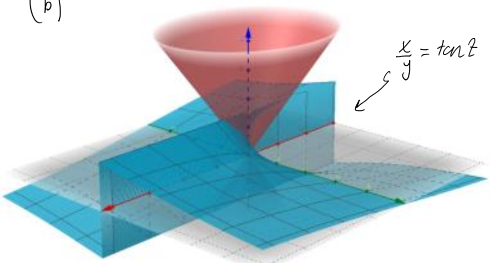
$= (3\cos t, 2\sin t, 6\cos t \sin t)$

with $t \in [0, 2\pi]$

(b)

cone

$\frac{x}{y} = \tan z$



Observe that

(i) $\frac{x}{y} = \tan z \Rightarrow x = y\tan z \quad (y \neq 0)$

Then

(ii) $z^2 = x^2 + y^2 = y^2\tan^2 z + y^2 = y^2(1+\tan^2 z) = \sec^2 z \cdot y^2$

So, $y^2 = \cos^2 z \cdot z$

So, $y = \mp \cos z \cdot z$

[the whole curve can not be parametrized by using single parametrization]

Define a parametrization as

$z = t$, $y = \cos t \cdot t$, and $x = y\tan(z) = t\cos(t)\tan(t)$

$(y \neq 0)$

where $t \in D \subseteq \mathbb{R}$ with $D = \mathbb{R} - \{0\} - \{(2k+1)\frac{\pi}{2} \mid k \in \mathbb{Z}\}$

Then we get a parametrization γ_1 for some part of the curve of intersection:

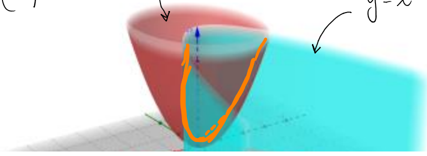
$\gamma_1(t) = (t\cos(t)\tan(t), t\cos(t), t)$, $t \in D$

For the other part, choose another param. γ_2, the same as γ_1, but $y = -t\cos(t)$. so that $\gamma = \gamma_1 \cup \gamma_2$.

(c)

$z = 4x^2 + y^2$

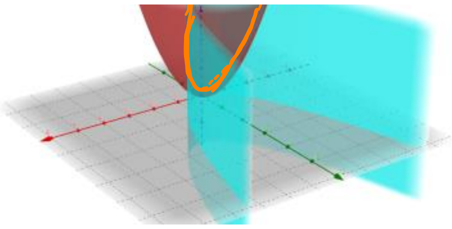
$y = x^2$



Let $x = t$. Then we get

$y = t^2$ and $z = 4x^2 + y^2 = 4t^2 + t^4$, $t \in \mathbb{R}$

The curve of intersection can be parametrized as



$y = t^2$ and $z = 4x^2 + y^2 = 4t + t$, $t \in \mathbb{R}$

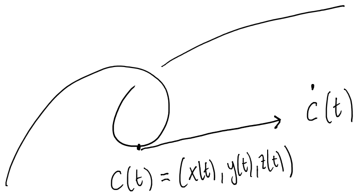
the curve of intersection can be parametrised as

$$\gamma(t) = \left(t,\ t^2,\ 4t^2 + t^4\right), \quad t \in \mathbb{R}$$

Question 02

Show that if the dot product of the velocity and acceleration of a moving particle is positive (or negative), then the speed of the particle is increasing (or decreasing).

soln



$$\dot{c}(t) = \langle \dot{x}(t), \dot{y}(t), \dot{z}(t) \rangle \quad \text{velocity.}$$

acceleration vector ; $\ddot{c}(t) = \langle \ddot{x}(t), \ddot{y}(t), \ddot{z}(t) \rangle$

By assumption,

$$\dot{c}(t) \cdot \ddot{c}(t) \geq 0 \quad (\text{or } \leq 0)$$

$$\Longleftrightarrow$$

$$\dot{x}\ddot{x} + \dot{y}\ddot{y} + \dot{z}\ddot{z} \geq 0 \quad (\text{or } \leq 0)$$

Let $s(t)$ be the speed func. at time t.

$$s(t) = \|\dot{c}(t)\| = \sqrt{\dot{x}^2(t) + \dot{y}^2(t) + \dot{z}^2(t)}, \quad \text{with } s(t) \neq 0 \ \forall t$$

Want to show $s(t)$ is $\nearrow$ whenever $\dot{c} \cdot \ddot{c} \geq 0$

(or $\searrow$ " " ≤ 0)

need to show $\dot{s}(t) \geq 0$ when $\dot{c} \cdot \ddot{c} \geq 0$

$(\leq 0$ " " $\leq 0)$

Observe that

$$\dot{s}(t) = \underbrace{\frac{1}{2\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}}}_{>0} \cdot \underbrace{\left(2\dot{x}\ddot{x} + 2\dot{y}\ddot{y} + 2\dot{z}\ddot{z}\right)}_{\text{by assump } \geq 0} \quad (\text{or } \leq 0)$$

So, $\dot{s}(t) \geq 0$ when $\dot{c} \cdot \ddot{c} \geq 0$

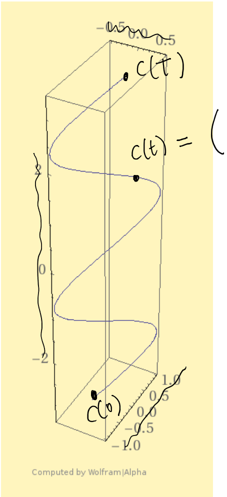
(or ≤ 0) $\quad (\leq 0)$

Question 03

Describe the parametric curve $\mathcal{C}$ given by

$$x = a\cos t\sin t,\ \ y = a\sin 2t,\ \ z = bt.$$

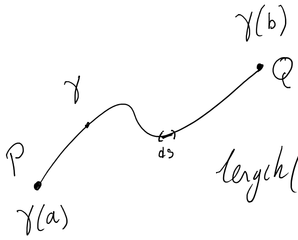
Express the length of $\mathcal{C}$ between $t = 0$ and $t = T > O$.



$$c(t) = (\overbrace{a\cos t\sin t}^{x},\ \overbrace{a\sin 2t}^{y},\ bt)$$

[notice : $y = 2x$]

Recall



$$\text{length}(\gamma) = \int_\gamma ds = \int_a^b \|\dot{\gamma}\|\, dt$$

In our case;

$$\dot{c}(t) = \langle -a\sin^2 t + a\cos^2 t,\ a\cos 2t \cdot 2,\ b\rangle$$

$$= \langle a\cos 2t,\ 2a\cos 2t,\ b\rangle$$

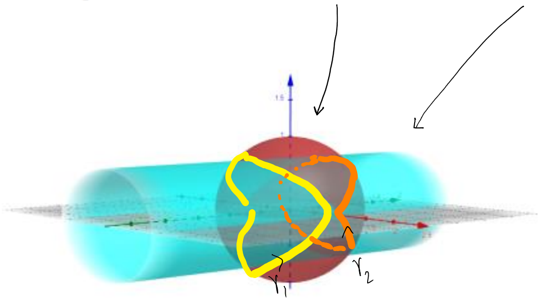
and so $\|\dot{c}(t)\| = \sqrt{5a^2\cos^2(2t) + b^2}$

Therefore,

$$\text{length}(C) = \int_0^T \sqrt{5a^2\cos^2(2t) + b^2}\, dt$$

Question 04

Describe the intersection of the sphere $x^2+y^2+z^2=1$ and the elliptic cylinder $x^2+2z^2=1$. Find the total length of this intersection curve.



$\underline{\underline{\gamma = \gamma_1 \cup \gamma_2}}$ the curve of intersection

Observe that

(1) Solving eqns at the same time gives

$$x^2+y^2+z^2=1 \iff \cancel{1}-2z^2+y^2+z^2=\cancel{1}$$

$$\iff \boxed{y=\mp z}$$

(2) Parametrize γ as follows:

As $x^2+\dfrac{z^2}{1/2}=1$, we let $x=\cos t$, $z=\sqrt{\tfrac{1}{2}}\sin t$ with $t\in[0,2\pi]$

and hence $y=\mp\sqrt{\tfrac{1}{2}}\sin t$.

(3) One part of the curve can be parametrized as

$$\gamma_1(t)=\left(\cos t,\ \sqrt{\tfrac{1}{2}}\sin t,\ \sqrt{\tfrac{1}{2}}\sin t\right),\quad t\in[0,2\pi]$$

and so, $\|\dot{\gamma}_1(t)\| = \left\|\left\langle -\sin t,\ \sqrt{\tfrac{1}{2}}\cos t,\ \sqrt{\tfrac{1}{2}}\cos t\right\rangle\right\| = 1$ //

The other part can be parametrized as:

$$\gamma_2(t)=\left(\cos t,\ -\sqrt{\tfrac{1}{2}}\sin t,\ \sqrt{\tfrac{1}{2}}\sin t\right),\quad \text{with } \|\dot{\gamma}_2\|=1$$ //

(4) $$\text{length}(\gamma) = 2\int_{\gamma_1} ds = 2\int_0^{2\pi} 1\,.\,dt = 4\pi$$ //

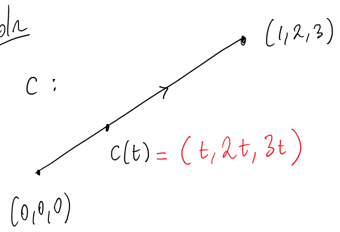
$\square$

Question 05

Find $\int_C x e^{yz}\, ds$ where C is the line segment from $(0,0,0)$ to $(1,2,3)$.

soln

C :



• Write down the eqn of C :

$$\langle x, y, z\rangle = \langle 0,0,0\rangle + t\langle 1,2,3\rangle, \quad t \in [0,1]$$

here $x = t$, $y = 2t$, and $z = 3t$. with $t \in [0,1]$.

• Compute ds :

$$ds = \| \dot{c}(t) \| \, dt = \| \langle 1,2,3\rangle \| \, dt = \sqrt{14}\, dt$$

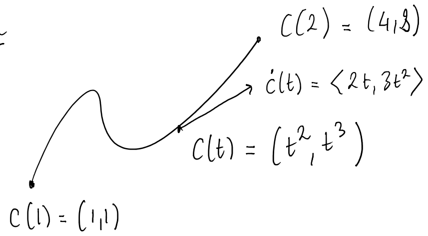
Then we have

$$\int_C x e^{yz}\, ds = \int_0^1 t\, e^{6t^2} \sqrt{14}\, dt \underset{\text{let } 6t^2 = u}{\overset{\cdots}{=}} \frac{\sqrt{14}}{12}\left(e^6 - 1\right) \quad //$$

Question 06

Evaluate $\int_C \frac{x^2}{y^{4/3}} ds$, where C is the curve $x = t^2, y = t^3$ for $1 \le t \le 2$

soln



(i) Compute ds :

$$ds = \| \dot{c}(t) \| \, dt = \| \langle 2t, 3t^2 \rangle \| \, dt = \sqrt{4t^2 + 9t^4} \, dt$$

(ii) Using the parametrization, we have

$$\int_C \frac{x^2}{y^{4/3}} ds = \int_1^2 \frac{\cancel{t^4}}{\cancel{t^4}} \sqrt{4t^2+9t^4} \, dt$$

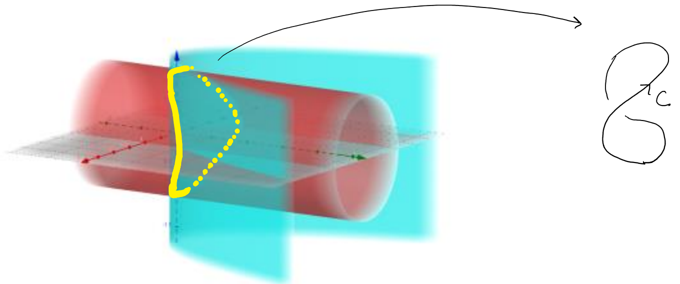
$$= \int_1^2 t \sqrt{4+9t^2} \, dt \quad (t \ge 0) \qquad \left(\text{let } 4+9t^2 = u \right)$$

$$\vdots$$

$$= \frac{1}{27} \left(40^{3/2} - 13^{3/2} \right)$$

Question 07

Find $\int_C \sqrt{1+4x^2z^2}\,ds$ where C is the curve of intersection of the surfaces $x^2+z^2=1$ and $y=x^2$.



(i) Parametrize C as follows:

$x = \cos(t)$, $z = \sin(t)$, and hence $y = \cos^2 t$

i.e. $C(t) = \left(\cos(t), \cos^2(t), \sin(t)\right)$, $0 \leq t \leq 2\pi$

(ii) $\dot{c}(t) = \langle -\sin(t), -\sin(2t), \cos(t) \rangle$

and so, $\|\dot{c}(t)\| = \sqrt{1+\sin^2(2t)}$

(iii) Compute:

$$\int_C \sqrt{1+4x^2z^2}\,ds = \int_0^{2\pi} \sqrt{1+4\cos^2(t)\sin^2 t} \cdot \sqrt{1+\sin^2(2t)}\,dt$$

$$\frac{1-\cos 4t}{2}$$

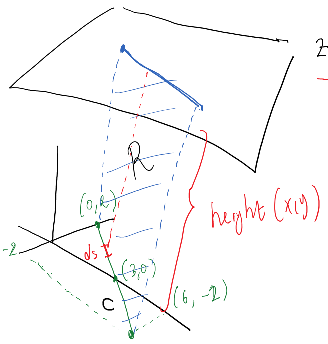
$$= \int_0^{2\pi} \left[1+\sin^2(2t)\right] dt$$

$\vdots$

$$= 3\pi$$

Question 08

Find the area of one side of "wall" standing orthogonally on the curve $2x + 3y = 6$, $0 \le x \le 6$ and beneath the curve on the surface $f(x, y) = 4 + 3x + 2y$.



$z = f(x,y) = 4 + 3x + 2y$

$$\text{Area}(R) = \int_C z \, ds$$

(i) Parametrize C as follows: $x = t$, $y = \dfrac{6-2t}{3}$ with $0 \le t \le 6$

then we get

$$c(t) = \left(t, \frac{6-2t}{3}\right) \quad \text{and} \quad \dot{c}(t) = \langle 1, -2/3 \rangle, \quad \|\dot{c}(t)\| = \sqrt{\tfrac{13}{9}}$$

(ii) Compute:

$$\int_C z \, ds = \int_C (4 + 3x + 2y) \, ds = \int_0^6 \left(4 + 3t + 2\,\frac{6-2t}{3}\right) \sqrt{\frac{13}{9}} \, dt \overset{\dots}{=} 26\sqrt{13}$$

Exercises

1. Evaluate $\int_C (x+y)ds$ where C is the straight line segment $x = t, y = 1-t, z = 0$, from $(0,1,0)$ to $(1,0,0)$

2. Evaluate $\int_C \sqrt{x^2+y^2}ds$ along the curve $\mathbf{r}(\mathbf{t}) = 4\cos t\mathbf{i} + 4\sin t\mathbf{j} + 3t\mathbf{k}, \quad -2\pi \leq t \leq 2\pi$

3. $\int_C zds$ along the part of the curve $x^2+y^2+z^2 = 1$, $x+y = 1$, where $z \geq 0$.