Calculus of Functions of Several Variables

Recitation-05: More on Double Integrals, and Triple Integrals

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Last time: Double integrals

Topics to be covered:

14.5 Triple Integrals **14.6 Change of Variables in Triple Integrals**	**14.5: 2,4,6,7,9,10,14,15** **14.6: 2,3,4,6,10,12,16**

Question 01

Evaluate the iterated integral $I = \int_0^3 \int_{\sqrt{x/3}}^1 e^{y^3}\,dy\,dx$. *(Hint: Change the order!)*

soln

Problem: It is hard to integrate wrt y first ∇

Trick: Change the order of iteration

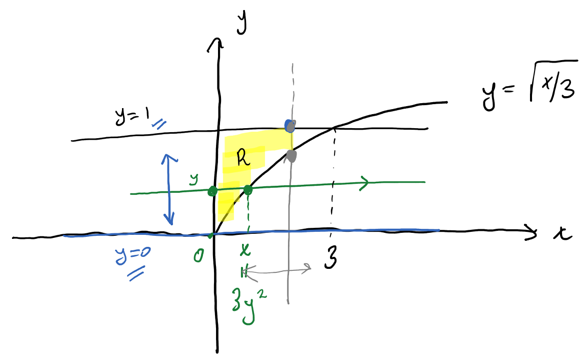
↳ • Check the boundries and reconstruct the region

• Write down the iterated integral in "dxdy"-order

We have

$$I = \int_0^3 \int_{\sqrt{x/3}\,=\,y}^{1\,=\,y} e^{y^3}\,dy\,dx$$

here, R:



starting with a vertical slice and then shift it horizontally

Now, we start with a horizontal slice, then shift it vertically.

$$I = \iint_R e^{y^3}\,dA = \int_0^1 \int_0^{3y^2} e^{y^3}\,dx\,dy$$

$$= \int_0^1 x\,e^{y^3}\Big|_0^{3y^2}\,dy$$

$$= \int_0^1 3y^2 e^{y^3}\,dy \qquad \left(\text{let } y^3 = u\right)$$

$$= e^{y^3}\Big|_0^1$$

$$= e - 1$$

$\square$

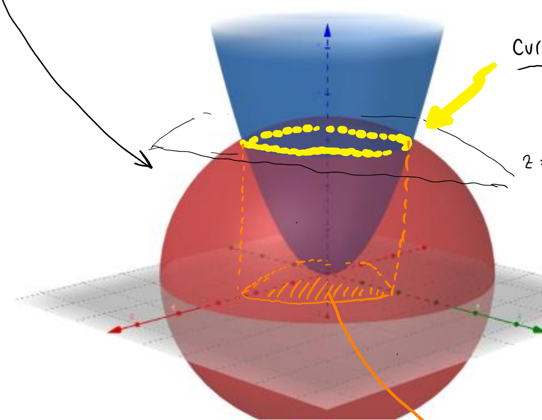
Question 02

Consider the space region Ω in the first octant (i.e. $x \geq 0, y \geq 0, z \geq 0$) which lies inside the sphere $x^2 + y^2 + z^2 = 12$ and above the paraboloid $z = x^2 + y^2$. Express the triple integral

$$\iiint_\Omega f(x, y, z)\mathrm{d}V$$

as an iterated triple integral

(a) in Cartesian coordinates (x, y, z).
(b) in Cylindrical coordinates (r, θ, z).
(c) in Spherical coordinates (ρ, ϕ, θ).



Curve of intersection : C $\quad 12 = x^2+y^2+z^2 = z + z^2$

$\Leftrightarrow$

$z^2 + z - 12 = 0$

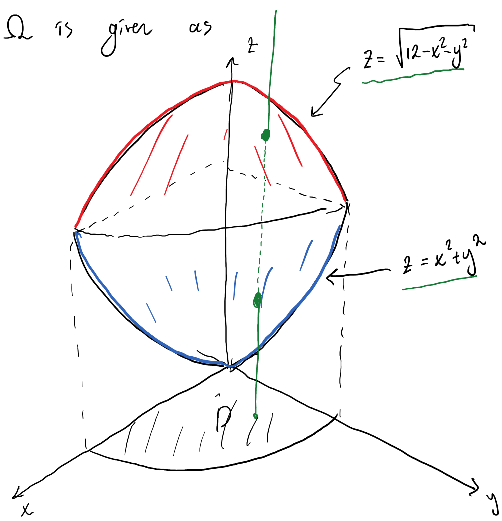
$\Leftrightarrow$

$\boxed{z = 3}$ or $z = -4$

so, $C: \quad x^2+y^2 = 3$ circle lies in the plane $z = 3$.

the base region D: $x^2+y^2 \leq 3$, $x, y \geq 0$

Then Ω is given as

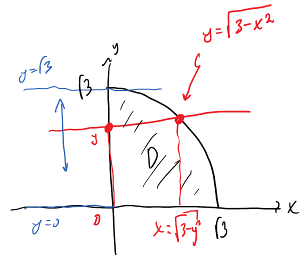


(a) $$I = \int_0^{\sqrt{3}} \int_0^{\sqrt{3-y^2}} \int_{x^2+y^2}^{\sqrt{12-x^2-y^2}} f(x,y,z) \; dz \; dx \; dy$$

start with a slice // z-axis then move it along the base region D

(using horizontal/vertical slices or rays as before)

where



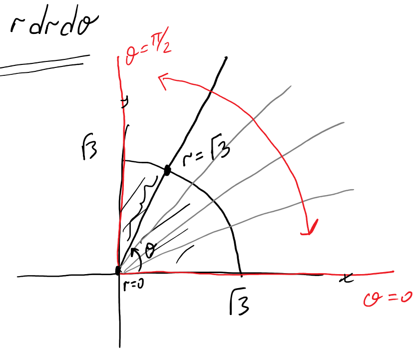
(b) let $x = r\cos\theta$, $y = r\sin\theta$, and $z = z$. Then we get

$z = x^2+y^2 = r^2$, $z = \sqrt{12-(x^2+y^2)} = \sqrt{12-r^2}$ and $dV = dz\, r\, dr\, d\theta$

so, we have

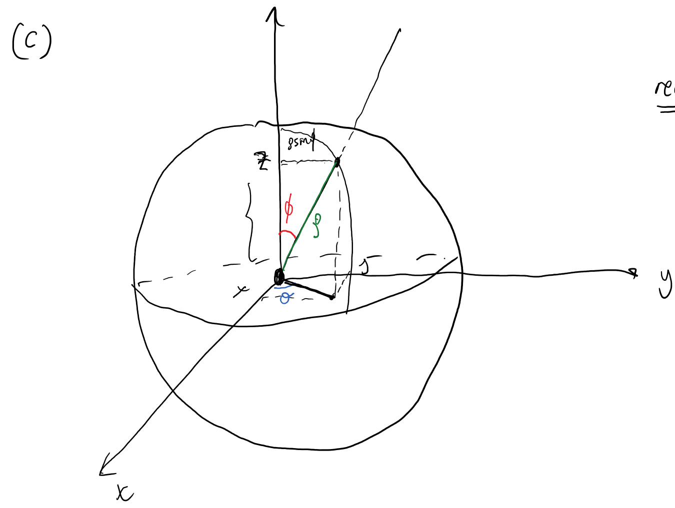
$$I = \int_0^{\pi/2} \int_0^{\sqrt{3}} \int_{r^2}^{\sqrt{12-r^2}} f(r,\theta,z) \; dz \; r\,dr\,d\theta$$

where



(c)

(c)



<u>recall</u> : $x = \rho \sin\phi \cos\theta$

$y = \rho \sin\phi \sin\theta$ with $dV = \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$

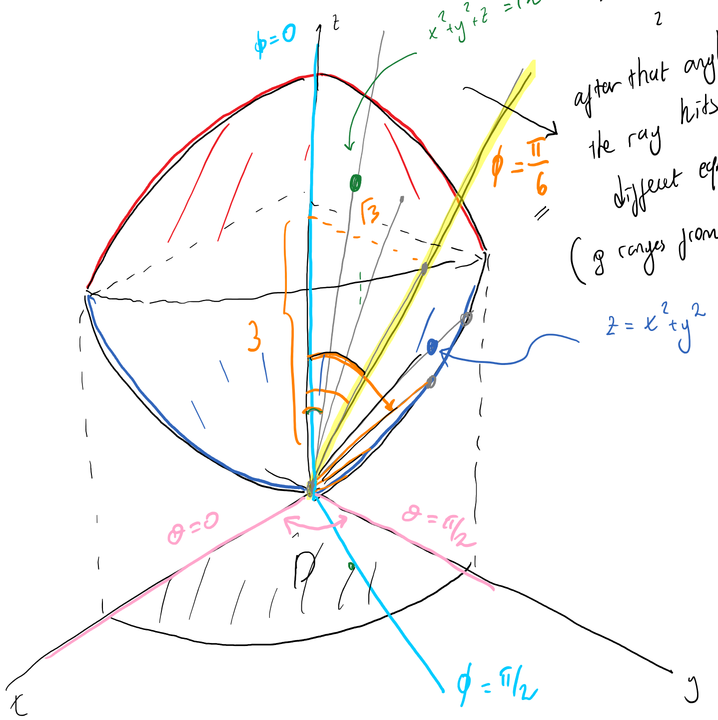
$z = \rho \cos\phi$

the volume element in (ρ, ϕ, θ)-system.

$x^2+y^2+z^2=12 \iff \underline{\underline{\rho = \sqrt{12}}}$ (ρ ranges from 0 to $\sqrt{12}$)

after that angle the point where the ray hits the region satisfies different eqn.

(ρ ranges from 0 $\cos\phi / \sin^2\phi$)



$z = x^2+y^2 \iff \rho\cos\phi = \rho^2 \sin^2\phi$

$\iff \rho(\rho\sin^2\phi - \cos\phi) = 0$

$\iff \rho = 0$ or $\rho = \cos\phi / \sin^2\phi$

so, we get

$$\int_0^{\pi/2}\int_0^{\pi/6}\int_0^{\sqrt{12}} f(\rho,\phi,\theta)\,\rho^2\sin\phi \, d\rho\, d\phi\, d\theta + \int_0^{\pi/2}\int_{\pi/6}^{\pi/2}\int_0^{\cos\phi/\sin^2\phi} f(\rho,\phi,\theta)\,\rho^2\sin\phi\, d\rho\, d\phi\, d\theta$$

Question 03

Consider the space region $\Omega = \{(x,y,z) \in \mathbb{R}^3 : z \geq \frac{1}{2},\ \ x^2+y^2+z^2 \leq 1\}$. Alternatively, Ω is the region lying inside the unit sphere $x^2+y^2+z^2=1$ and above the plane $z = 1/2$. Express the volume of region Ω

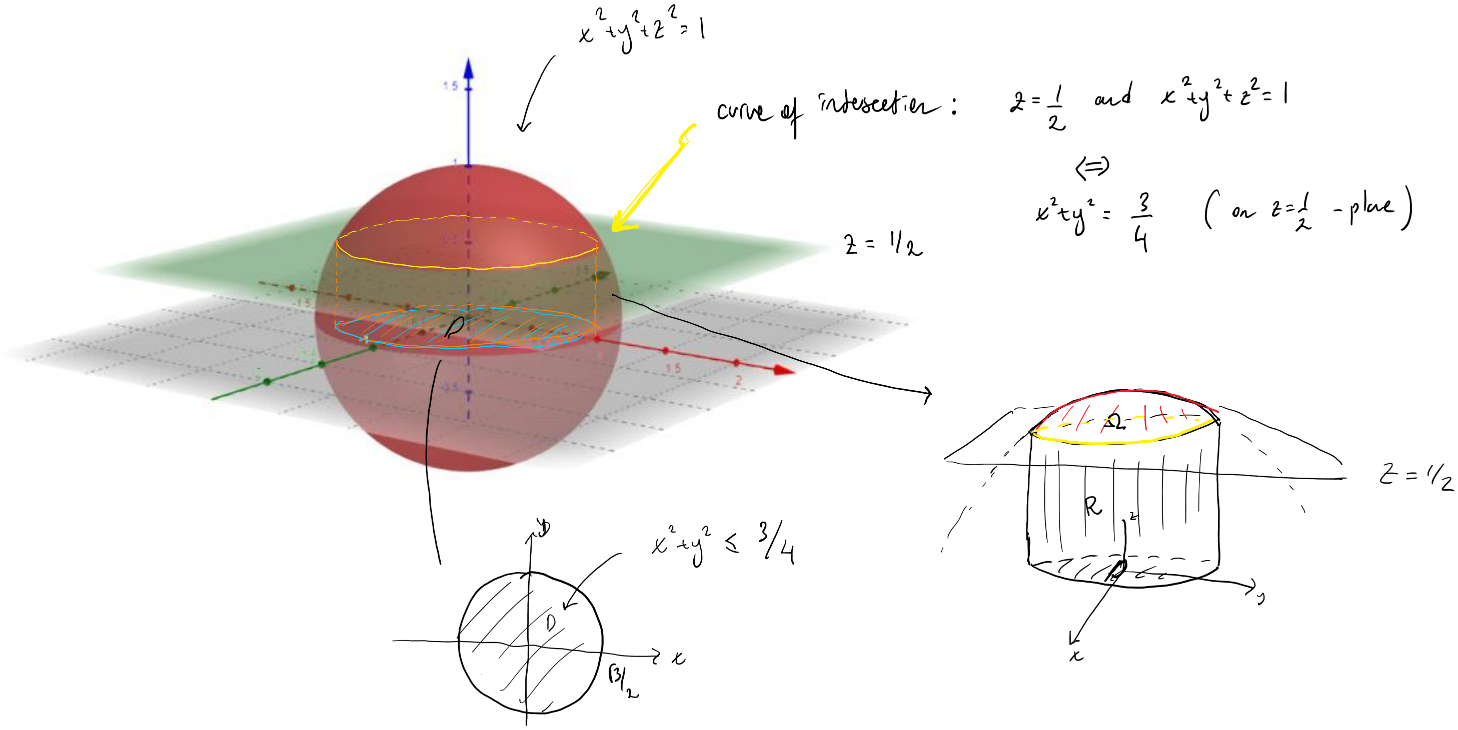
(a) as an iterated double integral in Cartesian coordinates (x, y).

(b) as an iterated double integral in Polar coordinates (r, θ).

(c) as an iterated triple integral in Cartesian coordinates (x, y, z).

(d) as an iterated triple integral in Cylindrical coordinates (r, θ, z).

(e) as an iterated triple integral in Spherical coordinates (ρ, ϕ, θ).



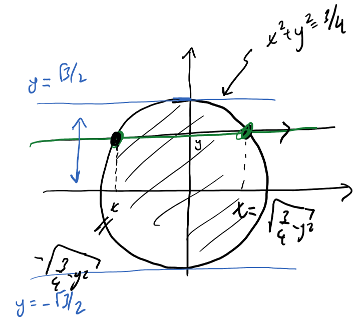
(a) $$V_\Omega = \underbrace{\iint_D \sqrt{1-(x^2+y^2)}\, dA}_{} - \underbrace{\iint_D \frac{1}{2}\, dA}_{}$$

volume of the region under the graph of the func. $z=\sqrt{1-x^2-y^2}$ on the region D.

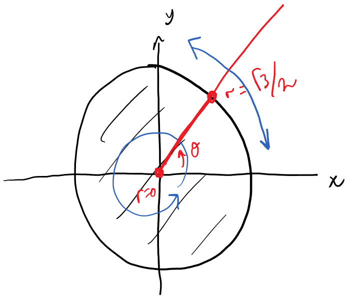
volume of R (∂R = cylinder with base circle $x^2+y^2 \leq 3/4$ and bdd above by $z=1/2$)

$$= \int_{-\sqrt{3}/2}^{\sqrt{3}/2} \int_{-\sqrt{\frac{3}{4}-y^2}}^{\sqrt{\frac{3}{4}-y^2}} \left[\sqrt{1-x^2-y^2} - \frac{1}{2} \right] dx\, dy$$

where

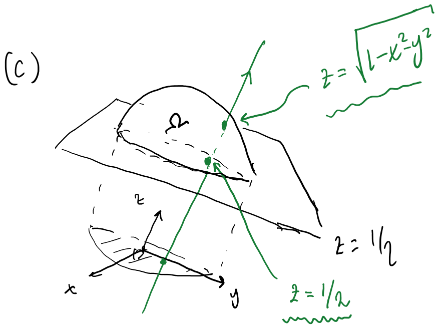


(b) Use $x = r\cos\theta$, $y = r\sin\theta$ (and hence $dA = r\,dr\,d\theta$)



$$V_\Omega = \int_0^{2\pi}\int_0^{\sqrt{3}/2}\left(\sqrt{1-r^2} - \frac{1}{2}\right) r\,dr\,d\theta$$

(c)



$$V_\Omega = \iiint_\Omega 1.\ dV$$

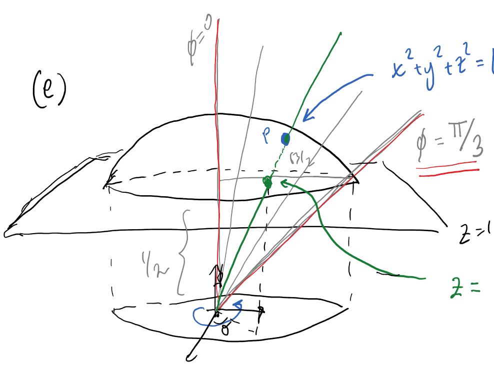
$$= \int_{-\frac{\sqrt{3}}{2}}^{\frac{\sqrt{3}}{2}} \int_{-\sqrt{\frac{3}{4}-y^2}}^{\sqrt{3/4-y^2}} \int_{1/2}^{\sqrt{1-x^2-y^2}} 1.\ dz\,dx\,dy$$

(d) Use (r, θ, z) where $x = r\cos\theta$, $y = r\sin\theta$, and $z = z$. with $dV = dz\, r\,dr\,d\theta$.

Then we have

$$V_\Omega = \iiint_\Omega 1.\,dV = \int_0^{2\pi}\int_0^{\sqrt{3}/2}\int_{1/2}^{\sqrt{1-r^2}} 1.\ dz\, r\,dr\,d\theta$$

(e)



$x^2+y^2+z^2 = 1 \iff \rho = 1$ (leaves the region)

here $x = \rho\sin\phi\cos\theta$
$y = \rho\sin\phi\sin\theta$ with $dV = \rho^2\sin\phi\,d\rho\,d\phi\,d\theta$
$z = \rho\cos\phi$

$z = 1/2 \iff \rho\cos\phi = 1/2$

$\iff \rho = \dfrac{1}{2\cos\phi}$. (hits the region)

so,
$$V_\Omega = \int^{2\pi}\int^{\pi/3}\int^{1} \rho^2\sin\phi\ d\rho\ d\phi\ d\theta$$

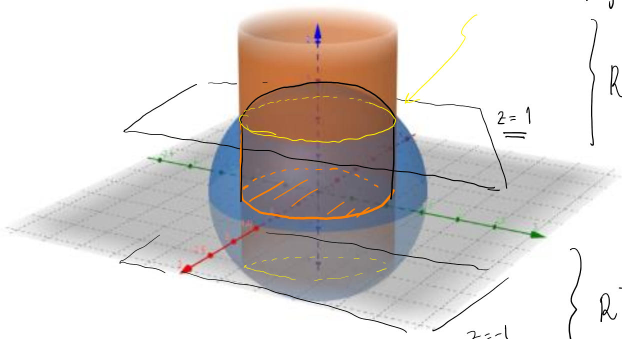
so,

$$V_{\Omega} = \int_0 \int_0 \int_{1/2\cos\phi} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

Question 04

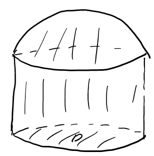
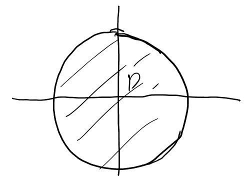
Let R be the region in $\mathbb{R}^3$ bounded by $x^2+y^2\le 1$ and $x^2+y^2+z^2\le 2$. Express the volume of R using an iterated (triple) integral in spherical coordinates. (Do **NOT** evaluate this integral.)

$x^2+y^2+z^2=2 \overset{x^2+y^2=1}{\iff} z^2=1 \iff z=\mp 1$

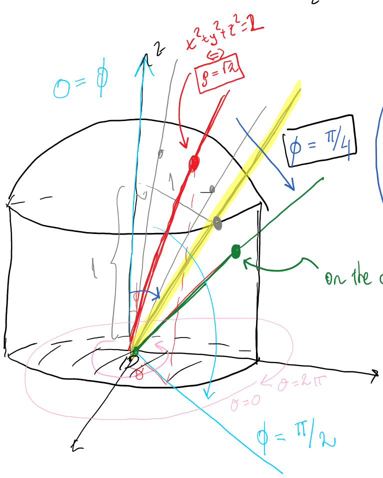


the curve(s) of intersection are circles $x^2+y^2=1$ lying on $z=1$ and $z=-1$ respectively.

Due to the symmetry, it is enough to study R^+.

R^+:  with 

D: $x^2+y^2\le 1$



after this angle the ray meets the surface at different types of points

let
$x = \rho\sin\phi\cos\theta$
$y = \rho\sin\phi\sin\theta$, with $dV = \rho^2\sin\phi\, d\rho\, d\phi\, d\theta$.
$z = \rho\cos\phi$

on the cylinder: $x^2+y^2=1 \iff \rho^2\sin^2\phi = 1$

$\iff \boxed{\rho = \frac{1}{\sin\phi}}$ $(\rho>0)$

In total, we get

$$V_R = \iiint_R 1.dV = 2\iiint_{R^+} 1.dV$$

$$= 2\left[\int_0^{2\pi}\int_0^{\pi/4}\int_0^{\sqrt{2}} \rho^2\sin\phi\, d\rho\, d\phi\, d\theta + \int_0^{2\pi}\int_{\pi/4}^{\pi/2}\int_0^{1/\sin\phi} \rho^2\sin\phi\, d\rho\, d\phi\, d\theta\right]$$

Exercises

1. Evaluate the iterated integral $I = \int_0^4 \int_{2x}^8 e^{x/y} \, dy dx$ *(Hint: Change the order!)*

2. Let R be the region in $\mathbb{R}^3$ obtained by removing the region defined by $x^2 + y^2 \leq 1$ from the region defined by $x^2 + y^2 + z^2 \leq 2$. Express the volume of R using an iterated (triple) integral in cylindrical coordinates. (Do **NOT** evaluate this integral.)

3. Let R be the region in $\mathbb{R}^3$ which is contained inside the sphere $x^2 + y^2 + z^2 = 2$ and which lies between the planes $z = 1$ and $z = -1$. Express the volume of R using iterated (triple) integrals in spherical coordinates. (Do **NOT** evaluate this integral.)