

Calculus of Functions of Several Variables

Recitation-04: Double Integrals

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Course webpage: <http://ma120.math.metu.edu.tr/>

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METU

Mathematics Department

Last time: Extreme values and the method of Lagrange multipliers

Topics to be covered:

Ch. 14: Multiple Integration

14.1 Double Integrals

14.2 Iteration of Double Integrals in Cartesian Coordinates

14.4 Double Integrals in Polar Coordinates

14.1: 5,13,15,18,19

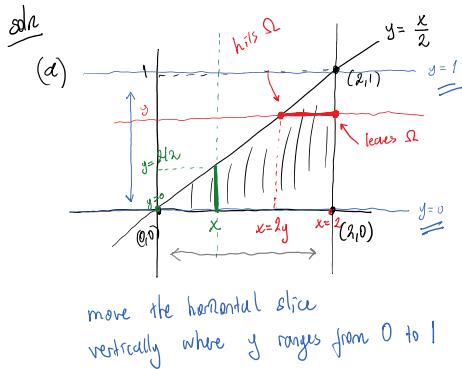
14.2: 1-27 odd

14.4: 1-25 odd

Question 01

Evaluate the following integrals by reducing them to suitable iterated integrals :

- $\iint_{\Omega} (x^2 + y^3) dA$ where Ω is the triangle with vertices $(0,0), (2,0), (2,1)$.
- $\iint_{\Omega} x^2 y^{1/3} dA$ where Ω is the finite region in the xy -plane bounded by the graphs of $y = x^2$ and $x = y^2$.



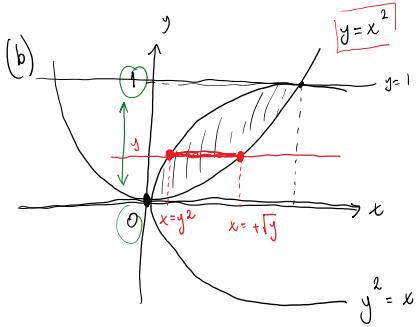
To compute $\iint_{\Omega} (x^2 + y^3) dA$, use the concept of iteration:

idea: "First integrate w.r.t one parameter while keeping the other one fixed."

$$\begin{aligned}\iint_{\Omega} (x^2 + y^3) dA &= \int_0^1 \left[\int_{2y}^x x^2 + y^3 \, dx \right] dy \\ &= \int_0^1 \frac{x^3}{3} + x y^3 \Big|_{2y}^x dy \\ &= \int_0^1 \frac{x^3}{3} + x y^3 - \frac{8y^3}{3} - 2y^4 \, dy \\ &\vdots \\ &= \text{exc}\end{aligned}$$

what if we choose the order "dy dx"

$$\int_0^2 \int_0^{x/2} x^2 y^3 \, dy \, dx$$



exc. compute the same double integral with the order "dy dx".

Then we have

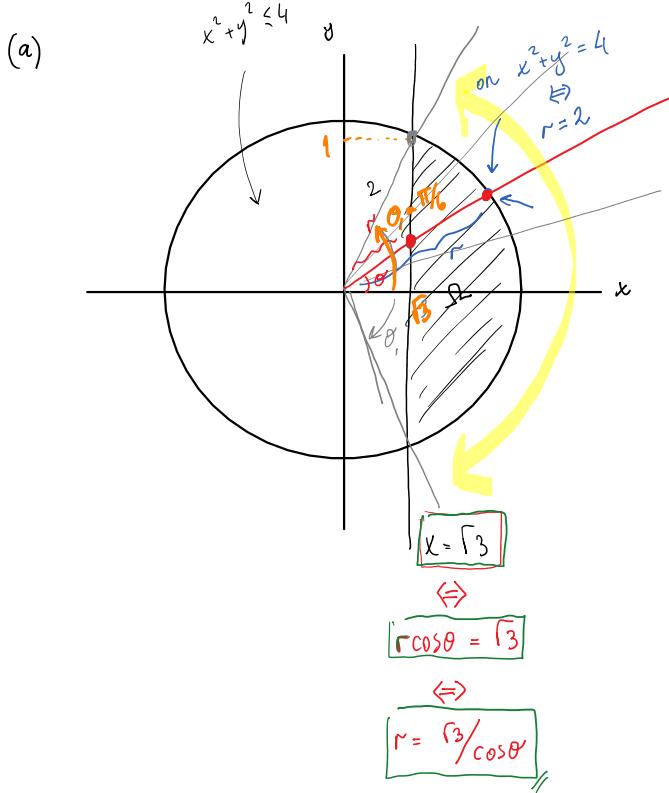
$$\begin{aligned}\iint_{\Omega} x^2 y^{1/3} dA &= \int_0^1 \int_{y^2}^{x^2} x^2 y^{1/3} \, dx \, dy \\ &= \int_0^1 \left(\frac{x^3}{3} y^{1/3} \Big|_{y^2}^{x^2} \right) dy \\ &= \frac{1}{3} \int_0^1 y^{11/6} - y^{10/3} \, dy \\ &\vdots \\ &= \frac{1}{3} \left(\frac{6}{17} - \frac{3}{11} \right)\end{aligned}$$

Question 02

Evaluate the following integrals by employing polar coordinates if necessary :

(a) $\iint_{\Omega} x \, dA$ where Ω is the region described by the inequalities $x^2 + y^2 \leq 4$ and $x \geq \sqrt{3}$.

(b) $\iint_{\Omega} xy \, dA$ where Ω is the region described by the inequality $x^2 + y^2 \leq 4x$.



Introduce polar coords : $x = r \cos \theta$, $y = r \sin \theta$ ($r \geq 0$)
($r\theta$ -system) with the area element $dA_{(r\theta)} = r dr d\theta$

Idea while iterating, instead of using horizontal/vertical slices, we now have "rays" with angles θ and length r .

so,

$$\iint_{\Omega} x \, dA = \int_{-\pi/6}^{\pi/6} \int_{\sqrt{3}/\cos \theta}^2 r \cos \theta \, r \, dr \, d\theta$$

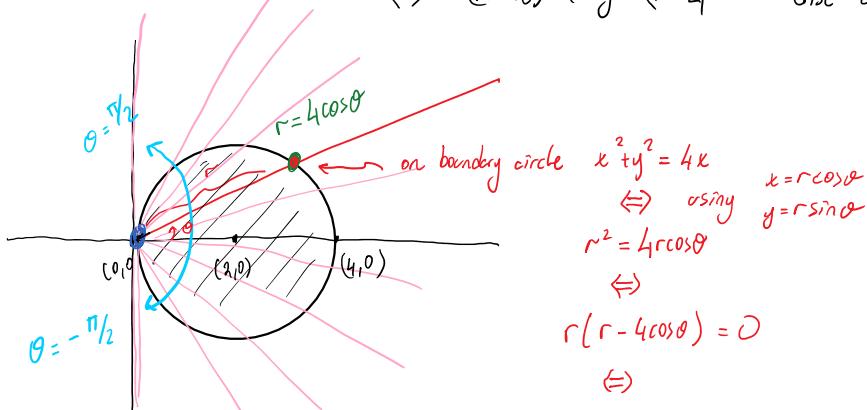
$$= \int_{-\pi/6}^{\pi/6} \left(\cos \theta \cdot \frac{r^3}{3} \Big|_{\sqrt{3}/\cos \theta}^2 \right) \, d\theta$$

$$= \int_{-\pi/6}^{\pi/6} \frac{8}{3} \cos \theta - \frac{2\sqrt{3}}{3} \cos \theta \cdot \frac{1}{\cos^3 \theta} \sec^2 \theta \, d\theta$$

$$= \frac{2}{3} \quad \underline{\text{check!}}$$

(b) Observe that $x^2 + y^2 \leq 4x \Leftrightarrow x^2 - 4x + y^2 \leq 0$

$$\Leftrightarrow (x-2)^2 + y^2 \leq 4 \quad \text{"disc centered at } (2,0) \text{ with } r=2 \text{ "}$$



$$\theta = -\frac{\pi}{2} \parallel$$

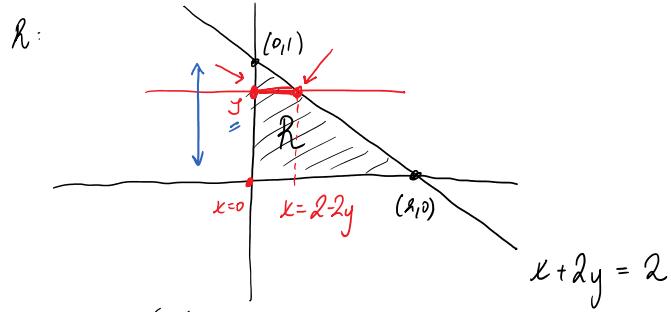
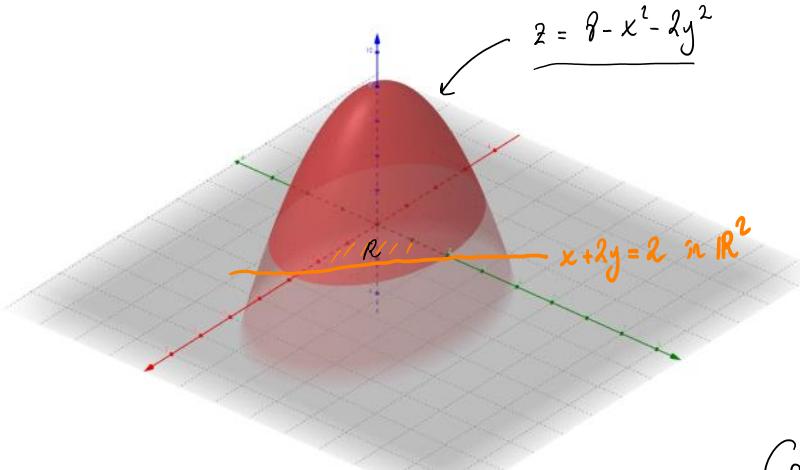
$$r(r - 4\cos\theta) = 0 \\ \Leftrightarrow \\ \underline{r=0} \text{ or } \underline{r=4\cos\theta}$$

Then we get, using polar coord,

$$\begin{aligned} \iint_{\Omega} xy \, dA &= \int_{-\pi/2}^{\pi/2} \int_0^{4\cos\theta} r^2 \sin\theta \cos\theta \, r \, dr \, d\theta \\ &= \int_{-\pi/2}^{\pi/2} \cos\theta \sin\theta \frac{r^4}{4} \Big|_0^{4\cos\theta} \, d\theta \\ &= 64 \int_{-\pi/2}^{\pi/2} \underbrace{\cos^5 \theta \sin \theta}_{\text{odd func.}} \, d\theta \\ &= 0 \end{aligned}$$

Question 03

Compute the volume of the space region under the graph of $z = 8 - x^2 - 2y^2$ that lies above the region in the xy -plane described by the inequalities $x \geq 0$, $y \geq 0$ and $x + 2y \leq 2$.



$$\text{Compute } V = \iint_R (8 - x^2 - 2y^2) \, dA.$$

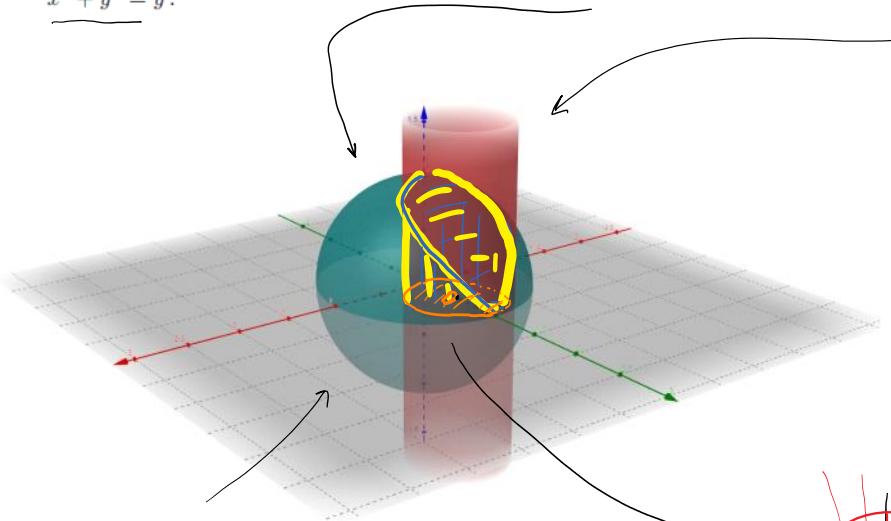
Using iteration, we have

$$\begin{aligned} V &= \int_0^1 \int_0^{2-2y} (8 - x^2 - 2y^2) \, dx \, dy \\ &= \int_0^1 \left[8x - \frac{x^3}{3} - 2xy^2 \right]_0^{2-2y} \, dy \\ &= \int_0^1 \left[8(2-2y) - \frac{(2-2y)^3}{3} - 2y^2(2-2y) \right] \, dy \\ &\quad \vdots \\ &= \underline{\underline{\text{exc}}}. \end{aligned}$$

□

Question 04

Compute the volume of the space region inside the sphere $x^2 + y^2 + z^2 = 1$ and inside the cylinder $x^2 + y^2 = y$.

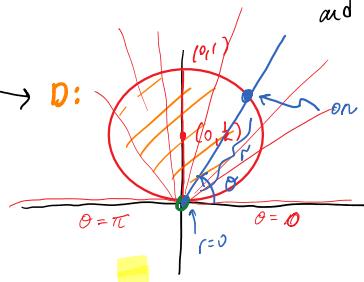


Observe that

$$x^2 + y^2 = y \Leftrightarrow x^2 + y^2 - y = 0$$

$$\Leftrightarrow x^2 + \left(y - \frac{1}{2}\right)^2 = \frac{1}{4}$$

circular cylinder
with base circle centered at $(0, \frac{1}{2}, 0)$
and $r = 1/2$



$$\begin{aligned} D: & \text{on } x^2 + y^2 = y \\ & \Leftrightarrow \text{using polar coord.: } x = r \cos \theta \\ & r^2 = r \sin \theta \\ & \Leftrightarrow r(r - \sin \theta) = 0 \\ & \Leftrightarrow r=0 \quad \text{or} \quad r=\sin \theta \end{aligned}$$

Volume is given by

$$V = 2 \iiint \sqrt{1-x^2-y^2} dA$$

$$= 2 \int_0^{\pi} \int_0^{\sin \theta} \sqrt{1-r^2} r dr d\theta$$

$$\begin{aligned} & \text{let } 1-r^2 = u \\ & \text{then } -2r dr = du \end{aligned}$$

$$= - \int_0^{\pi} \int_1^{\cos^2 \theta} u^{1/2} du d\theta$$

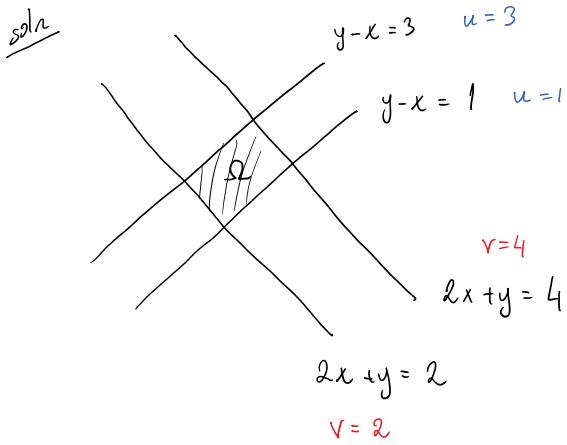
$$= - \int_0^{\pi} \frac{2}{3} u^{3/2} \Big|_1^{\cos^2 \theta} d\theta$$

$$\begin{aligned} &= \frac{2}{3} \int_0^{\pi} (1 - \cos^3(\theta)) d\theta \\ &\text{hint: } \cos \theta \cdot \cos^2 \theta \\ &= \frac{2}{3} \pi \end{aligned}$$

$$= \frac{2}{3} \pi$$

Question 05

Evaluate the double integral $\iint_{\Omega} (x^2 + y^2) dA$ where Ω is the region described by the inequalities $1 \leq y - x \leq 3$ and $2 \leq 2x + y \leq 4$.



Hard to use the standard iterations in xy -coordinates.

so, we change the coordinates:

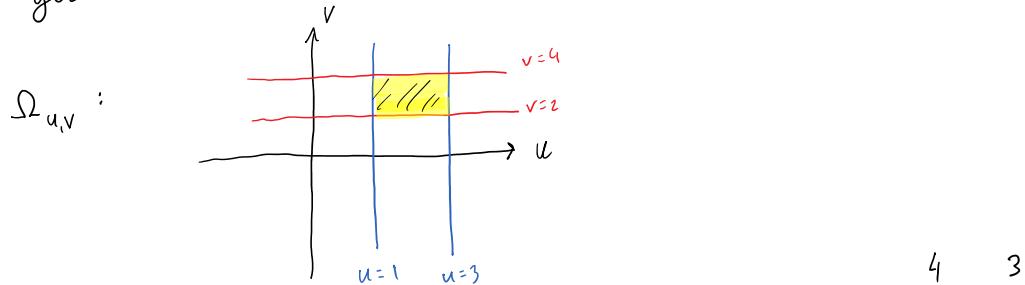
let $y-x =: u$ and $2x+y =: v$. Then $x = \frac{v-u}{3}$,

the area element becomes

$$y = \frac{2u+v}{3}$$

$$\begin{aligned} dA_{(u,v)} &= \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du dv \\ &= \left| \det \begin{bmatrix} -1/3 & 1/3 \\ 2/3 & 1/3 \end{bmatrix} \right| du dv \\ &= \frac{1}{3} du dv \end{aligned}$$

and so we get



so,

$$\begin{aligned} \iint_{\Omega} x^2 + y^2 dA &= \iint_{\Omega_{uv}} \left[\left(\frac{v-u}{3} \right)^2 + \left(\frac{2u+v}{3} \right)^2 \right] \frac{1}{3} du dv \\ &= \int_2^4 \int_1^3 \dots \frac{1}{3} du dv \\ &= \dots \\ &\text{exc.} \end{aligned}$$

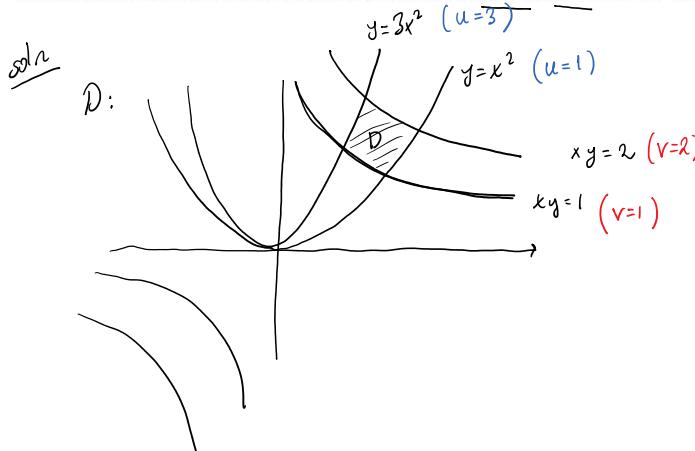
using
 uv -coordinates

Question 06

Evaluate the double integral

$$\mathcal{I} := \iint_{\mathcal{D}} \frac{6}{x} y^2 \sin(x^2 y^2) dA$$

for the plane region \mathcal{D} which is bounded by the curves $xy = 1$, $xy = 2$, $y = x^2$ and $y = 3x^2$.



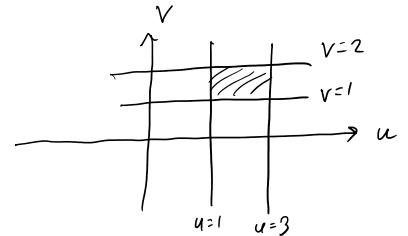
want to introduce new and more practical coordinates:

$$\text{let } u := \frac{y}{x^2} \quad \text{and} \quad v = xy. \quad \text{Then}$$

$$(i) \text{ Jacobian: } \frac{\partial(u,v)}{\partial(x,y)} = \det \begin{bmatrix} -\frac{2y}{x^3} & \frac{1}{x^2} \\ y & x \end{bmatrix} = -\frac{2y}{x^2}$$

$$\text{and hence } \frac{\partial(x,y)}{\partial(u,v)} = -\frac{x^2}{2y}$$

(ii) Region becomes:



with the area element

$$dA_{(u,v)} = \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du dv = \frac{x^2}{2y} du dv \quad (y>0)$$

Compute:

$$\iint_{\mathcal{D}} 6 \frac{y^2}{x} \sin(x^2 y^2) dA \underset{\substack{\text{uv-coordinates} \\ \downarrow}}{=} \iint_{\mathcal{D}_{(u,v)}} 6 \frac{y^2}{x} \sin(v^2) \frac{x}{2y} du dv$$

$$\mathcal{D}_{(u,v)}: 2 \leq v \leq 3$$

$$= 2 \int_1^2 \int_1^3 v \sin(v^2) du dv$$

$$= 2 \int_1^2 u v \sin(v^2) \Big|_1^3 dv$$

$$= 2 \int_1^2 2 v \sin(v^2) dv \quad \left[\text{let } v^2 = t \right]$$

:

exercise

Exercises

- (a) Evaluate $\iint_{\Omega} \ln(y) dA$ where Ω is the finite region in the first quadrant bounded by the graphs of $xy = 1$, $x = 3$ and $y = 2$.
- (b) Evaluate $\iint_{\Omega} y dA$ where Ω is the region described by the inequalities $r \leq 1 + \cos \theta$ and $y \geq 0$.
- (c) Evaluate the double integral $\mathcal{I} := \iint_{\mathcal{D}} \left(1 + \frac{3y^2}{x^2}\right)^2 \cos(x^2 + 3y^2) dA$ for the plane region \mathcal{D} in the **first** quadrant bounded by the curves $y = x$, $y = 2x$, $x^2 + 3y^2 = 1$ and $x^2 + 3y^2 = 3$.