Calculus of Functions of Several Variables
Recitation-03

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Last time: Linear approximations, gradients, directional derivatives, and implicit functions

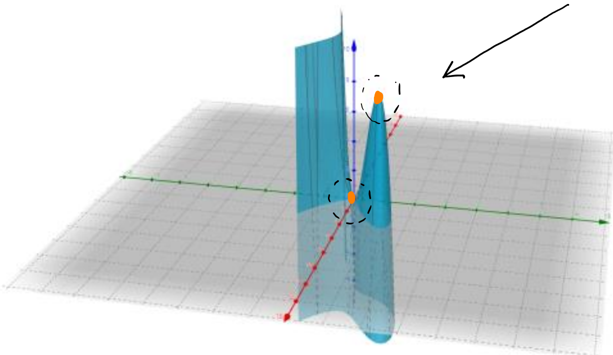
Topics to be covered:

Ch. 13: Applications of Partial Derivatives	
13.1 Extreme Values	**13.1: 1, 3, 6, 7, 9, 11, 17, 19, 24, 26**
13.2 Extreme Values of Functions Defined on Restricted Domains	**13.2: 3, 5, 7, 8, 9, 11, 17**
13.3 Lagrange Multipliers	**13.3: 1, 3, 5, 7, 9, 11, 19, 21, 22**

Question-01

Find and classify the critical points of the function

$$f(x,y) = 3y^2 - 2y^3 - 3x^2 + 6xy.$$



$\nabla f(p) = \langle 0,0 \rangle$

soln Recall: To classify critical points p of f we need to check the determinant of the Hessian of f:

$$H_f(p) = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}\Bigg|_p , \quad \det H_f(p) = f_{xx}f_{yy} - f_{xy}f_{yx}\Big|_p$$

Thm

1) $\det H_f(p) > 0$ and $f_{xx}(p) > 0 \Rightarrow f$ has a local min at p

2) $\det H_f(p) > 0$ and $f_{xx}(p) < 0 \Rightarrow$ " " " max at p

3) $\det H_f(p) < 0 \Rightarrow p$ is a saddle point

4) $\det H_f(p) = 0 \Rightarrow$ no info.

• Compute $f_x = -6x + 6y$, $f_y = 6y - 6y^2 + 6x$

$f_{xx} = -6$, $f_{xy} = 6$, $f_{yy} = 6 - 12y$, $f_{yx} = 6$

$\Rightarrow$ get $\boxed{H_f(p) = \begin{bmatrix} -6 & 6 \\ 6 & 6-12y \end{bmatrix}}$

• $\nabla f(x,y) = \langle 0,0 \rangle$ if (i) $6y - 6x = 0$

$\langle f_x, f_y \rangle \big|_{p=(x,y)}$ (ii) $6y - 6y^2 + 6x = 0$

Observe that (i) gives $\boxed{y = x}$ and hence eqn (ii) gives

$6y - 6y^2 + 6y = 0 \iff y^2 - 2y = 0$

$\iff y = 0$ or $y = 2$

$(x=0)$ $(x=2)$

so, the critical points: $P_1 = (0,0)$, $P_2 = (2,2)$

For the classification, check $\det H_f(P_i)$ for each i:

• $\det H_f(0,0) = \det \begin{bmatrix} -6 & 6 \\ 6 & 6 \end{bmatrix} = -72 < 0$

So, $P_1 = (0,0)$ is a saddle point.

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- $\det H_f(2,2) = \det \begin{bmatrix} -6 & 6 \\ 6 & -18 \end{bmatrix} > 0$ and $f_{xx}(2,2) < 0$

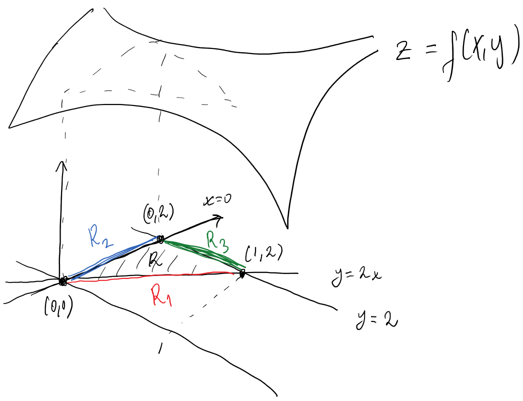
So, f has a local max at $(2,2)$

Question-02

Find the absolute maximum and minimum values of the function

$$f(x, y) = 2x^2 - 4x + y^2 - 4y + 1$$

on the triangular region bounded by $x = 0, y = 2, y = 2x$.



• Existence: Note f is cont. everywhere (since f is a poly.)
In particular, f is cont. on closed and bdd region R,
By EVT, f has abs. max/min

• How to locate them

By EVT, enough look at the local data or boundaries.
(critical pts / singular pts — interior points —)

(i) look at $\nabla f(x,y) = \langle 4x-4, 2y-4 \rangle$

so, • $\nabla f(x,y) = \langle 0,0 \rangle$ where $(x,y) = (1,2)$ not in the interior of R

• f_x and f_y are defined everywhere, so no singular points.

(ii) Check boundaries: $\partial R = R_1 \cup R_2 \cup R_3$.

on R_1: $y = 2x$, $0 \leq x \leq 1$ Then we get

$$f(x,y)\Big|_{y=2x} = 6x^2 - 12x + 1 =: g(x)$$

Here, $g(x)$ on $[0,1]$ is cont. func. on a closed and bdd domain, so by EVT, it has abs max/min.

To locate them:

– check $g'(x) = 12x - 12 = 0$ if $x = 1$ (not a interior pt)

– end points: $g(0) = 1$
$g(1) = -5$

$\Rightarrow$ $\boxed{\begin{aligned} f(0,0) &= 1 \\ f(1,2) &= -5 \end{aligned}}$

on R_2: $x = 0$, $0 \leq y \leq 2$: $f(x,y)\Big|_{\text{on } R_2} = y^2 - 4y + 1 =: h(y)$, again cont on $[0,2]$.

so, $h(y)$ has abs. max/min.

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- look at $h'(y) = 2y-4=0$ wherever $y=2$
- " $h(0)=1$, $\underline{h(2)} = -3$

$\Rightarrow$ $f(0,0) = 1$, $\boxed{f(0,2) = -3}$

on R_3: $y=2$, $0 \leq x \leq 1$: $\underset{\text{on } R_3}{f(x,y)} = 2x^2-4x-3 =: k(x)$ on $[0,1]$

Clearly, $k(x)$ is cont on $[0,1]$, so, $k(x)$ has abs max/min. by EVT.

To locate them:

- look at $k'(x) = 4x-4 = 0$ if $x=1$
- " $k(0) = -3$ and $k(1) = -5$

$\Rightarrow$ $f(0,2) = 3$, $f(1,2) = -5$

To sum up, checking those values gives

f has an abs max at $(0,0)$, $f(0,0)=1$

f " " min at $(1,2)$, $f(1,2) = -5$.

Question-03

Find the absolute maximum and minimum values of the function

$$f(x,y,z) = 2x + 3y + z$$

subject to the unit sphere $x^2 + y^2 + z^2 = 1$. $\left(g(x,y,z) = 0 \right)$

soln idea: to understand global beh. of f, study its Lagrangian func. and its local behavior.

Use the method of Lag. multiplers;

let $\underbrace{L(x,y,z,\lambda) = f(x,y,z) + \lambda g(x,y,z)}_{\text{Lagrangian function of } f}$ where $g(x,y,z) = x^2+y^2+z^2-1$

here, $L(x,y,z,\lambda) = 2x + 3y + z + \lambda(x^2+y^2+z^2-1)$, λ: Lagrange multiplier.

Look at the points P for which $\nabla L \Big|_P = \langle 0,0,0,0 \rangle$ (*)

i.e

$$0 = \frac{\partial L}{\partial x} = 2 + 2\lambda x$$

$$0 = \frac{\partial L}{\partial y} = 3 + 2\lambda y$$

$$0 = \frac{\partial L}{\partial z} = 1 + 2\lambda z$$

$\Longrightarrow$ $x = -1/\lambda$, $y = -3/2\lambda$, $z = -1/2\lambda$

$$0 = \frac{\partial L}{\partial \lambda} = x^2+y^2+z^2-1$$

substitute into the last eqn

$$\frac{1}{\lambda^2} + \frac{9}{4\lambda^2} + \frac{1}{4\lambda^2} - 1 = 0$$

$\Longleftrightarrow$

$$\frac{14 - 4\lambda^2}{4\lambda^2} = 0 \quad \text{with } \lambda \neq 0$$

$4\lambda^2$

which means $\boxed{\lambda = \mp \sqrt{7}/2}$

(for $\lambda = \sqrt{7}/2$)

so, the desired points are $P_1 = (x, y, z) = \left(-\frac{2}{\sqrt{7}}, -\frac{3}{\sqrt{7}}, -\frac{1}{\sqrt{7}}\right)$

$P_2 = \left(\frac{2}{\sqrt{7}}, \frac{3}{\sqrt{7}}, \frac{1}{\sqrt{7}}\right)$ for $\lambda = -\sqrt{7}/2$

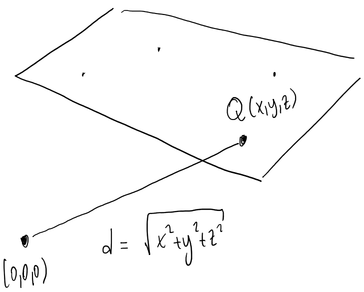
observe $f(P_1) = -14/\sqrt{7}$ "abs min"

✓

$f(P_2) = 14/\sqrt{7}$ "abs max"

Question-04

Using the Lagrange multiplier method, find the point Q on the plane $\mathcal{P}: x + 2y + 2z = 3$ that is closest to the origin.



$\mathcal{P}: \; x + 2y + 2z = 3$

It is enough to understand the extreme values of the func.

$$S(x,y,z) = x^2 + y^2 + z^2$$

subject to the constraint $x + 2y + 2z = 3$ $\quad \left[\underbrace{x + 2y + 2z - 3}_{g(x,y,z)} = 0 \right]$

Use the method of Lag. multipls:

Define the Lagrangian func. for S by

$$L(x,y,z,\lambda) = \underbrace{x^2 + y^2 + z^2}_{S} + \lambda \big(\underbrace{x + 2y + 2z - 3}_{g} \big)$$

Then look at the critical points of L:

$$\nabla L \Big|_{(x,y,z,\lambda)} = \langle 0, 0, 0, 0 \rangle \quad \text{if}$$

$$\left.\begin{aligned} \frac{\partial L}{\partial x} &= 2x + \lambda = 0 \\ \frac{\partial L}{\partial y} &= 2y + 2\lambda = 0 \\ \frac{\partial L}{\partial z} &= 2z + 2\lambda = 0 \end{aligned}\right] \; (*)$$

$$\frac{\partial L}{\partial \lambda} = x + 2y + 2z - 3 = 0$$

Observe that

(*) implies that $\boxed{x = -\lambda/2, \; y = -\lambda, \text{ and } z = -\lambda.}$

Using the last eqn gives $\boxed{\lambda = -2/3}$

the point we need is $Q = \left(1/3, \; 2/3, \; 2/3 \right)$

Exercises

1. Find the absolute maximum and minimum values of the function

$$f(x, y) = x^4 + y^2$$

on the unit disk $D = \{(x, y) : x^2 + y^2 \leq 1\}$. (Use the Lagrange multiplier method on ∂D.)

2. Let $f(x, y) = x^3 y^5$ and $g(x, y) = x + y$.

   (a) Find the absolute maximum of f subject to the constraint $g(x, y) = 8$ using the Lagrange multiplier method.

   (b) By restricting f onto the line $x+y = 8$, verify that the value obtained in the first part is indeed the absolute maximum of f, and also that f does not have any absolute minimum value on $x+y = 8$.