Last time: Functions of several variables, limits/cont., partial deriv., higher-order deriv., the chain rule.

Topics to be covered:

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Find the indicated derivatives assuming that the function $f(x, y)$ has continuous partial derivatives.

(a) $\frac{\partial}{\partial x} f(y^4, x^3)$

(b) $\frac{\partial}{\partial x} f(x^2 f(x, t), f(y, t))$

(a) Let $u(x, y) = y^4$ and $v(x, y) = x^3$. Then we have $z = f(u, v)$.

Using the chain rule,

$$\frac{\partial f}{\partial x} = f_1 \frac{\partial u}{\partial x} + f_2 \frac{\partial v}{\partial x} = f_2 \cdot 3x^2$$

(b) Look at $f(x^2 f(x, t), f(y, t))$, $u = x^2 f(x, t)$, $v = f(y, t)$

Then

$$\frac{\partial}{\partial x} f(x^2 f(x, t), f(y, t)) = f'_1 \left[ 2x f(x, t) + x^2 f'_1(x, t) \right]$$

$$= f'_1(u, v) \cdot \left[ 2x f(x, t) + x^2 f'_1(x, t) \right]$$
Find \( \frac{\partial^2}{\partial y \partial x} f(y^2, xy, -x^2) \) in terms of partial derivatives of \( f \).

Let \( u(x,y) = y^a, \ v(x,y) = xy \) and \( w(x,y) = -x^2 \). Then we have

**Compute:**

\[
\frac{\partial}{\partial y} \left[ \frac{\partial f}{\partial x} \right] = \frac{\partial}{\partial y} \left[ f_1 \cdot \frac{\partial f}{\partial x} + f_2 \cdot \frac{\partial v}{\partial x} + f_3 \cdot \frac{\partial w}{\partial x} \right]
\]

\[
= \frac{\partial}{\partial y} \left[ y \cdot f_2 \right] - \frac{\partial}{\partial y} \left[ 2x \cdot f_3 \right], \quad \text{where}
\]

\[
= 1 \cdot f_2 + y \left[ f_{21} \frac{\partial u}{\partial y} + f_{22} \frac{\partial v}{\partial y} + f_{23} \frac{\partial w}{\partial y} \right]
\]

\[
- 2x \left[ f_{31} \frac{\partial u}{\partial y} + f_{32} \frac{\partial v}{\partial y} + f_{33} \frac{\partial w}{\partial y} \right]
\]

\[
= f_2 + y \left[ 2y f_{21} + x f_{22} \right] - 2x \left[ 2y f_{31} + x f_{32} \right]
\]
Let \( f(x, y) = \begin{cases} 
\frac{x^2 - y^2}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\
0 & \text{if } (x, y) = (0, 0) 
\end{cases} \)

(a) Find \( f_x(0,0) \) and \( f_y(0,0) \).
(b) Show that \( f_{yy}(0,0) = 1 \) and \( f_{yy}(0,0) = -1 \).
(c) Does part (b) contradict the fact \( f_{xx}(0,0) = f_{yy}(0,0) \)?

\[
\begin{align*}
\text{[a]} \quad f_x(0,0) &= \lim_{h \to 0} \frac{f(0+h, 0) - f(0,0)}{h} \\
&= \lim_{h \to 0} \frac{0/h^2 - 0}{h} = 0 \\
f_y(0,0) &= \lim_{h \to 0} \frac{f(0, 0+h) - f(0,0)}{h} \\
&= \lim_{h \to 0} \frac{0/h^2 - 0}{h} = 0
\end{align*}
\]

(b) First observe that \( (x, y) \neq (0,0) \) we have

\[
\begin{align*}
\frac{f_x(x,y)}{x^2+y^2} = \frac{y (x^4 + 4x^2y^2 - y^4)}{(x^2+y^2)^2} \quad \text{and} \quad f_y(x,y) = -\frac{x(y^4 + 4x^2y^2 - x^4)}{(x^2+y^2)^2}
\end{align*}
\]

From part (a), we conclude that

\[
G(x,y) = \begin{cases} \frac{y (x^4 + 4x^2y^2 - y^4)}{(x^2+y^2)^2} & \text{if } (x, y) \neq (0,0) \\
0 & \text{if } (x, y) = (0,0) \end{cases}
\]

\[
H(x,y) = \begin{cases} -\frac{x(y^4 + 4x^2y^2 - x^4)}{(x^2+y^2)^2} & \text{if } (x, y) \neq (0,0) \\
0 & \text{if } (x, y) = (0,0) \end{cases}
\]

So, want to compute \( G_y(0,0) \) and \( H_x(0,0) \).
\[ (\Rightarrow f_{xy}(0,0) = f_{yx}(0,0) = \lim_{h \to 0} \frac{G(0, 0 + h) - G(0, 0)}{h} = \lim_{h \to 0} \frac{-h^5/h^4 - 0}{h} = -1 \]

\[ (\Rightarrow f_{xy}(0,0) = f_{yx}(0,0) = \lim_{k \to 0} \frac{H(0+k, 0) - H(0, 0)}{k} = \lim_{k \to 0} \frac{k^5/k^4 - 0}{k} = 1 \]

as desired.

(c) No contradiction \( \Rightarrow \) the hypothesis of the theorem fails
\[ \rightarrow \quad \text{f and all partials should be cont. at } (0,0) \]

But, in fact, one can show that
\[ f_{xy} \text{ and } f_{yx} \text{ are not cont. at } (0,0) \]

Computations \( \Rightarrow \) (Check)

\[ f_{xy}(x, y) = \begin{cases} 
(x^4 + 12x^2y^2 - 5y^4)(x^2y^2) + y(x^4 + 4x^2y^2 - y^4) + 2(x^2y^2)2y \\
(x^2y^2)^4 \\
-1
\end{cases} \]

if \( (x, y) \neq (0,0) \)

if \( (x, y) = (0,0) \)

Check that \( \lim_{(x, y) \to (0,0)} f_{xy}(x, y) \) d.n.e.

\( \Rightarrow \) enough to check \( \lim_{(x, y) \to (0,0)} f_{xy}(x, y) \) and \( \lim_{(x, y) \to (0,0)} f_{yx}(x, y) \)

on \( \Gamma_1 : x = 0 \)

on \( \Gamma_2 : y = 0 \)
and conclude that they are not the same

( this would show that \( \lim_{(x,y) \to (a,b)} f_{xy}(x,y) \) d.n.e. )
Use a suitable linearization to approximate the number $\sqrt{0.99 \ e^{0.02}}$

so, recall (one-variable case)

\[
\frac{f(x) - f(a)}{x - a} = \text{slope} = f'(a)
\]

whence

\[
L(x) = f'(a)(x-a) + f(a)
\]

\[
f(x) \approx L(x)
\]

"a linearization of f at x=a"

\[
\text{can be generalized to higher dims.}
\]

instead of "tangent line,
we use the tangent plane $P$ at $(a,b)$

to give a linear of f.

the linearization of $f$ near $(a,b)$ is

\[
L(x,y) = f(a,b)(x-a) + f_x(a,b)(y-b) + f(a,b)
\]

For the approximation, write $f(x,y) \approx L(x,y)$

Let $f(x,y) = \sqrt{x^2 + y^2}$. To approximate the nb $\sqrt{0.99 \ e^{0.02}} (= f(0.99, 0.02))$

we the linearization of $f$ at $(1,0)$

\[
of_x = \frac{y}{2 \sqrt{y^2}} \ e^{y/2}, \quad of_y = \frac{x}{2 \sqrt{y^2}} \ e^{y/2}, \quad \text{and} \quad of_x(1,0) = \frac{1}{2}, \quad of_y(1,0) = 1, \quad f(1,0) = 1
\]

so,

\[
of(x,y) \approx L(x,y) = \frac{1}{2} (x-1) + 1 \cdot (y-0) + 1
\]

so,

\[
f(0.99, 0.02) \approx \frac{1}{2} (-0.01) + 0.02 + 1 = 1.015
\]

\[
\therefore \sqrt{0.99 \ e^{0.02}} \approx 1.015
\]
Consider the function \( f(x, y, z) = \frac{x}{y} - z \) at \( P_0(3, 1, 1) \)

(a) Compute \( \nabla f(P_0) \)

(b) Is there a unit vector \( \mathbf{u} \) such that \( D_\mathbf{u} f(P_0) = 5 \)? If yes, find one; if no, prove that it does not exist.

(c) Is there a unit vector \( \mathbf{u} \) such that \( D_\mathbf{u} f(P_0) = 3 \)? If yes, find one; if no, prove that it does not exist.

(d) Let \( S \) be a set of all points \( P(x, y, z) \) where \( f \) increases fastest in the direction of the vector \( \mathbf{A} = \langle 2, 1, 2 \rangle \). Describe the set \( S \).

\[
\nabla f(P_0) = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle \bigg|_{P_0} = \left\langle \frac{1}{y}, -\frac{x}{y^2}, -1 \right\rangle = \left\langle 1, -3, -1 \right\rangle
\]

(b) Since \( f \) is differentiable at \( P_0 \) (check why), for any unit vector \( \mathbf{u} \),

\[D_\mathbf{u} f(P_0) = \mathbf{u} \cdot \nabla f(P_0)\]

Also recall that the max rate of change is in the direction of \( \nabla f(P_0) \) with magnitude \( |\nabla f(P_0)| \).

The min rate of change is in the direction of \( -\nabla f(P_0) \) with magnitude \( -|\nabla f(P_0)| \).

Here we have

\[-|\nabla f(P_0)| \leq D_\mathbf{u} f(P_0) \leq |\nabla f(P_0)| = \sqrt{11} \quad (< \ 5)\]

So, \( \exists \) a vector \( \mathbf{u} \) such that \( D_\mathbf{u} f(P_0) = 5 \)

(c) \( 3 \in [-\sqrt{11}, \sqrt{11}] \) \( \implies \) It is possible to find \( \mathbf{u} \) with \( ||\mathbf{u}|| = 1 \) s.t. \( D_\mathbf{u} f(P_0) = 3 \)

\[
i \quad 3 = \langle u_1, u_2, u_3 \rangle \cdot \langle 1, -3, -1 \rangle \iff u_1 - 3u_2 - u_3 = 3 \quad (\text{with } u_1^2 + u_2^2 + u_3^2 = 1)
\]

Take \( \mathbf{u} = \langle 0, -1, 1 \rangle \)

(d) Recall: \( f \) increases fastest in the direction of \( \nabla f(P_0) \).

So, the set \( S \) is given by

\[
S = \left\{ P = (x, y, z) \in \text{Domain} \mid \nabla f(P) \parallel \mathbf{A} = \langle 2, 1, 2 \rangle \right\}
\]

\[
L \rightarrow \nabla f(P) = k \cdot \mathbf{A}
\]

Just rescale \( \mathbf{A} \) by scalar \( k = \frac{1}{2} \)

\[
\langle -1, -\frac{1}{2}, -\frac{1}{2} \rangle = \nabla f(P) = \left\langle \frac{1}{y}, -\frac{x}{y^2}, -1 \right\rangle
\]

\( \therefore \)
\[
\begin{align*}
&y = 1, \\
&\frac{1}{y} = -1, \quad -\frac{x}{y^2} = -\frac{1}{2} \\
\iff
&y = -1, \quad x = \frac{1}{2}, \quad z = t, \quad t \in \mathbb{R}
\end{align*}
\]

That means
\[
S = \left\{ \left( \frac{1}{2}, -1, t \right) : t \in \mathbb{R} \right\}
\]
defines a line \( l \) with an eqn:
\[
\langle x, y, z \rangle = \langle \frac{1}{2}, -1, 0 \rangle + t \langle 0, 0, 1 \rangle, \quad t \in \mathbb{R}
\]
Suppose that you are climbing a hill whose shape is given by the equation \( z = 1000 - 0.01x^2 - 0.02y^2 \) and you are standing at a point with coordinates \((90, 100, 760)\).

(a) In which direction should you proceed initially in order to reach the top of the hill faster?
(b) If you climb in that direction, at what angle above the horizontal will you be climbing initially?

\[
\begin{align*}
\text{(a) the same rate of change can be attained in the direction } & \nabla z (P) \text{ where} \\
\nabla z (P) &= \begin{pmatrix} \frac{\partial z}{\partial x} \\ \frac{\partial z}{\partial y} \end{pmatrix} |_{P} \\
&= \begin{pmatrix} -2x \\ -4y \end{pmatrix} \\
&= \begin{pmatrix} -180 \\ -400 \end{pmatrix} \\
\text{hence } & \nabla z (P) \text{ by } \vec{a} = \begin{pmatrix} -180 \\ -400 \end{pmatrix}
\end{align*}
\]

(b) Harder than it seems: \( \vec{a} \) here is to determine the angle \( \theta \):

Write down the equation of the line \( L \) passing through the point \((90, 100, 760)\) in the direction of the vector \( \vec{a} \):

\[
\vec{r} = \begin{pmatrix} 90 \\ 100 \\ 760 \end{pmatrix} + t \begin{pmatrix} -180 \\ -400 \end{pmatrix}
\]

or \( x = 90 - 180t \), \( y = 100 - 400t \), and \( z = 760 \).

Write \( z \)-direction as "free", then the set \( \{ (x,y,z): x=90-180t, y=100-400t \} \)

defines a plane \( \mathcal{P} \) as in the picture above. Notice that the intersection of the plane \( \mathcal{P} \) with the \( xy \)-plane is the line \( L \).

Now consider the intersection of the surface \( S \) and the plane \( \mathcal{P} \).

Let \( C \) be the curve of intersection given by the parametrization

\[
c(t) = \begin{pmatrix} 90 - 180t \\ 100 - 400t \\ 760 + (180t)(-400t) - (90)(100 - 400t) \end{pmatrix}
\]

where \( P_t \in C \) and \( \vec{a}_t = \vec{c}(t) = \begin{pmatrix} 180 \\ 400 \\ 760 \end{pmatrix} \).

Having these observations, [part-b] asks for the angle \( \theta \) between the vectors \( \vec{c}(t) \) and \( \vec{a} \) (i.e., we wish to compute the angle between \( \vec{c}(t) \) and \( \nabla z (P) \)).

\[
\text{Compute:
}
\]

\[
\vec{c}(0) = \begin{pmatrix} 90 \\ -40 \end{pmatrix} - \begin{pmatrix} 90(180t) \\ 100(400t) \end{pmatrix} = \begin{pmatrix} -180t \\ -400t \end{pmatrix}
\]

and hence \( || \vec{c}(0) || = \sqrt{180^2 + 400^2} \)

\[
\text{(ii) } || \vec{a} || = \sqrt{(-180)^2 + (-400)^2} = 510.44
\]

Thus we have

\[
\cos (\theta) = \frac{\vec{c}(0) \cdot \vec{a}}{|| \vec{c}(0) || \ || \vec{a} ||} = \frac{180 \cdot -180}{510.44 \cdot 510.44} = \frac{1}{102.44}
\]

\[
\theta = \arccos \left( \frac{1}{102.44} \right)
\]
Find the equation of the tangent plane to the level surface of the function $f(x,y,z) = \frac{xy}{z^2}$ at the point $P = (1,2,3)$.

At $t = f(x,y,z) = \frac{xy}{z^2}$, for each fixed $t = t_0$, $t_0 = f(x_0, y_0, z_0)$ goes an inside of surface.

At $P = (1,2,3)$, the level surface is

$$\frac{xy}{z^2} = \frac{2}{3} \left( = f(1,2,3) \right)$$

Compute $\nabla f(P) = \left< \frac{y}{z^2}, \frac{x}{z^2}, -\frac{2xy}{z^3} \right> \bigg|_{(1,2,3)}$

$$= \left< \frac{2}{3}, \frac{1}{3}, -\frac{4}{27} \right>$$

Since $\nabla f(P) \parallel \vec{n}$, just take $\vec{n} = \left< 6, 3, -4 \right>$.

At $(1,2,3)$ is given by

$$6(x-1) + 3(y-2) - 4(z-3) = 0$$

$$\iff$$

$$6x + 3y - 4z = 0$$
Check that near the point \((1,0)\) the equation
\[\sin xy + y \ln x + e^{xy} - 1 = 0\]
can be solved for \(y\) as a function of \(x\) and find the value of \(\frac{dy}{dx}\) at the given point.

**Solu**

\[\text{Let } F(x,y) = \sin xy + y \ln x + e^{xy} - 1 \text{ with the eqn } F(x,y) = 0.\]

It is enough to check

(i) \(F_x \mid_{(1,0)} \neq 0\) (ii) All partial deriv. exist and are cont at \((1,0)\).

Observe \(F_x \Bigg|_{(1,0)} = \left(\frac{\cos(xy)+ \ln(x) + xe^{xy}}{xy} \right)\bigg|_{(1,0)} = 2 \neq 0\)

So, given eqn. can be solved for \(y\) as a func. of \(x\), say \(y = f(x)\) with \(f(1) = 0\).

To find \(\frac{dy}{dx}\), observe that

\[
\begin{align*}
F &= y - x \\
\text{Taking deriv. of both sides of } F = 0 \text{ wrt } x \text{ gives } \\
F_x(x,y) + F_y(x,y) \cdot \frac{dy}{dx} &= 0
\end{align*}
\]

So,
\[
\frac{dy}{dx} = - \frac{F_x(x,y)}{F_y(x,y)} \bigg|_{(1,0)} = - \frac{\cos(xy) + \frac{y}{x} + ye^{xy}}{\cos(xy) + \ln(x) + xe^{xy}} \bigg|_{(1,0)} = 0
\]