Calculus of Functions of Several Variables
Recitation-02: Linear approximations, gradients, directional derivatives, and implicit functions

Kadri İlker Berktav
berktav@metu.edu.tr

Course webpage: http://ma120.math.metu.edu.tr/

Department of Mathematics:
http://math.metu.edu.tr/

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Last time: Functions of several variables, limits/cont., partial deriv., higher-order deriv., the chain rule

Topics to be covered:

12.6 Linear Approximations	**12.6: 4,6,10,16**
12.7 Gradients and Directional Derivatives	**12.7: 4,8,10,17,18,19,22,26,36**
12.8 Implicit Functions *("Systems of Equations" is not included)*	**12.8: 2,5,6,11**

Question-01

Find the indicated derivatives assuming that the function $f(x, y)$ has continuous partial derivatives.

(a) $\frac{\partial}{\partial x} f(y^4, x^3)$

(b) $\frac{\partial}{\partial x} f(x^2 f(x,t), f(y,t))$

(a) Let $u(x,y) = y^4$ and $v(x,y) = x^3$. Then we have $z = f(u,v)$

$u - [x, y$

$v - [x, y$

Using the chain rule,

$$\frac{\partial f}{\partial x} = f_1 \frac{\partial u}{\partial x} + f_2 \frac{\partial v}{\partial x} = f_2 \cdot 3x^2$$

(b) look at $f(\underbrace{x^2 f(x,t)}_{u}, \underbrace{f(y,t)}_{v})$, $u = x^2 f(x,t)$, $v = f(y,t)$

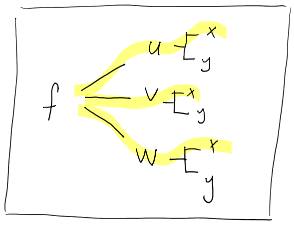
Then

$$\frac{\partial}{\partial x} f(x^2 f(x,t), f(y,t)) = f_1 \cdot \left[2x f(x,t) + x^2 f_1(x,t) \right]$$

$$= f_1(u,v) \cdot \left[2x f(x,t) + x^2 f_1(x,t) \right]$$

Question-02

Find $\frac{\partial^2}{\partial y \partial x} f(y^2, xy, -x^2)$ in terms of partial derivatives of f.

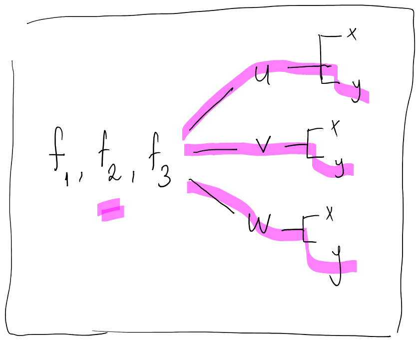


<u>soln</u> Let $u(x,y) = y^2$, $v(x,y) = xy$ and $w(x,y) = -x^2$. Then we have

Compute:

$$\frac{\partial}{\partial y}\left[\frac{\partial f}{\partial x}\right] = \frac{\partial}{\partial y}\left[f_1 \cdot \overset{0}{\cancel{\frac{\partial u}{\partial x}}} + f_2 \overset{y}{\frac{\partial v}{\partial x}} + f_3 \cdot \overset{-2x}{\frac{\partial w}{\partial x}} \right]$$

$$= \frac{\partial}{\partial y}\left[y \cdot f_2 \right] - \frac{\partial}{\partial y}\left[2x \cdot f_3 \right] \quad , \text{ where}$$



$$= 1 \cdot f_2 + y\left[f_{21} \frac{\partial u}{\partial y} + f_{22} \frac{\partial v}{\partial y} + f_{23} \frac{\partial w}{\partial y} \right]$$

$$- 2x \left[f_{31} \frac{\partial u}{\partial y} + f_{32} \frac{\partial v}{\partial y} + f_{33} \frac{\partial w}{\partial y} \right]$$

$$= f_2 + y\left[2y f_{21} + x f_{22} \right] - 2x \left[2y f_{31} + x f_{32} \right]$$

Question-03

Let $f(x,y) = \begin{cases} \dfrac{x^3y - xy^3}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$ //

(a) Find $f_x(0,0)$ and $f_y(0,0)$.

(b) Show that $f_{xy}(0,0) = -1$ and $f_{yx}(0,0) = 1$.

(c) Does part (b) contradict the fact $f_{xy}(0,0) = f_{yx}(0,0)$?

we are interested in the change in the first component.

soln

(a)
$$f_x(0,0) = \lim_{h \to 0} \frac{f(0+h,0) - f(0,0)}{h}$$
$$= \lim_{h \to 0} \frac{0/h^2 - 0}{h} = 0$$
$$f_y(0,0) = \lim_{h \to 0} \frac{f(0,0+h) - f(0,0)}{h}$$
$$= \lim_{h \to 0} \frac{0/h^2 - 0}{h} = 0$$

(b) First observe that, for $(x,y) \neq (0,0)$ we have

$$f_x(x,y) = \frac{y(x^4 + 4x^2y^2 - y^4)}{(x^2+y^2)^2} \quad // \quad \text{and} \quad f_y(x,y) = -\frac{x(y^4 + 4x^2y^2 - x^4)}{(x^2+y^2)^2} \quad //$$

From part(a), we conclude that

$$f_x(x,y) = \begin{cases} \dfrac{y(x^4+4x^2y^2-y^4)}{(x^2+y^2)^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

$=: G(x,y)$

$$f_y(x,y) = \begin{cases} -\dfrac{x(y^4+4x^2y^2-x^4)}{(x^2+y^2)^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

$=: H(x,y)$

so, want to compute $G_y(0,0)$ and $H_x(0,0)$.

$(\;\; f \;(0,0))$ $\qquad$ $(= f \;(0,0))$

$(= f_{xy}(0,0))$

(i) $\underline{\underline{f_{xy}(0,0)}} = \underline{\underline{G_y(0,0)}} = \lim\limits_{h\to 0} \dfrac{G(0,0+h) - G(0,0)}{h}$

$= \lim\limits_{h\to 0} \dfrac{-h^5/h^4 - 0}{h} = -1$

(ii) $\underline{\underline{f_{yx}(0,0)}} = \underline{\underline{H_x(0,0)}} = \lim\limits_{k\to 0} \dfrac{H(0+k,0) - H(0,0)}{k}$

$= \lim\limits_{k\to 0} \dfrac{k^5/k^4 - 0}{k} = 1$

as desired.

(c) $\underline{\underline{\text{No}}}$ contradiction ! the hypothesis of the thm $\underline{\underline{\text{fails}}}$

$\hookrightarrow$ f and $\underline{\text{all partials}}$ should be cont. at $(0,0)$
$(f_1, f_2, f_{12}, f_{21})$

$\underline{\underline{\text{But}}}$ in fact one can show that

f_{xy} and f_{yx} are $\underline{\text{not}}$ cont. at $(0,0)$

Computations $\Rightarrow$ (Check !)

$$f_{xy}(x,y) = \begin{cases} \dfrac{(x^4 + 12x^2y^2 - 5y^4)(x^2+y^2)^2 + y(x^4 + 4x^2y^2 - y^4) \cdot 2(x^2+y^2)\,2y}{(x^2+y^2)^4} & \text{if } (x,y) \neq (0,0) \\ -1 & \text{if } (x,y) = (0,0) \end{cases}$$

Check that $\lim\limits_{(x,y)\to(0,0)} f_{xy}(x,y)$ d.n.e.

$\hookrightarrow$ enough to check $\lim\limits_{\substack{(x,y)\to(0,0) \\ \text{on } \Gamma_1:\, x=0}} f_{xy}(x,y)$ and $\lim\limits_{\substack{(x,y)\to(0,0) \\ \text{on } \Gamma_2:\, y=0}} f_{xy}(x,y)$

and conclude that they are <u>not</u> the same

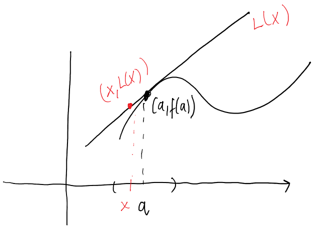
(this would show that $\lim_{(x,y)\to(0,0)} f_{xy}(x,y)$ d.n.e..)

Question-04

Use a suitable linearization to approximate the number $\sqrt{0{,}99}\, e^{0{,}02}$

soln

recall (one-variable case)



$$\frac{L(x)-f(a)}{x-a} = \text{slope} = f'(a)$$

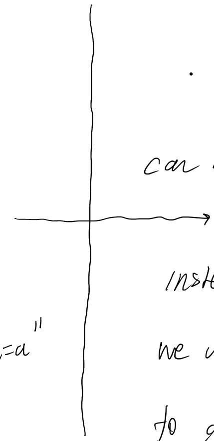
when exists

$$\Rightarrow \boxed{L(x) = f'(a)(x-a) + f(a)}$$

Approximation of f for x near a

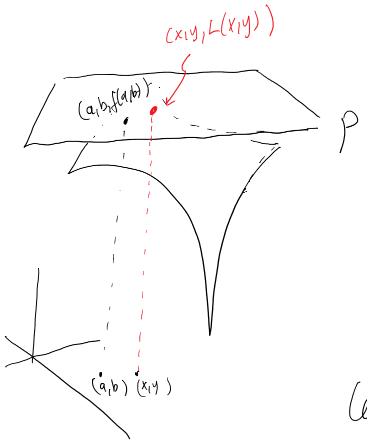
$f(x) \approx L(x)$

"a linearization of f at x=a"



can be generalized to higher dim's.

Instead of "tangent line", we use the tangent plane P at (a,b) to give a linear of f.



the linearization of f near (a,b) is

$$(z=)\ \boxed{L(x,y) = f_1(a,b)(x-a) + f_2(a,b)(y-b) + f(a,b)}$$

For the approximation, write $f(x,y) \approx L(x,y)$

Let $f(x,y) = \sqrt{x}\, e^y$. To approximate the nb $\sqrt{0{,}99}\, e^{0{,}02}$ $\left(= f(0{,}99, 0{,}02)\right)$ use the linearization of f at (1,0)

Observe: $f_x = \frac{1}{2\sqrt{x}} e^y$, $f_y = \sqrt{x}\, e^y$, and $f_x(1,0) = \frac{1}{2}$, $f_y(1,0) = 1$, $f(1,0) = 1$

So,

$$f(x,y) \approx L(x,y) = \frac{1}{2}(x-1) + 1\cdot(y-0) + 1$$

So,

$$f(0.99, 0.02) \approx \frac{1}{2}(-0{,}01) + 0{,}02 + 1 = 1{,}015$$ //

$$\therefore \sqrt{0.99}\, e^{0.02} \approx 1{,}015$$

☐

Question-05

Consider the function $f(x,y,z) = \frac{x}{y} - \ z$ at $P_0(3,1,1)$

(a) Compute $\nabla f(P_0)$

(b) Is there a unit vector $\mathbf{u}$ such that $D_{\mathbf{u}} f(P_0) = 5$. If yes find one, if no prove that it does not exist.

(c) Is there a unit vector $\mathbf{u}$ such that $D_{\mathbf{u}} f(P_0) = 3$. If yes find one, if no prove that it does not exist.

(d) Let S be a set of all points $P(x,y,z)$ where f increases fastest in the direction of the vector $\mathbf{A} = \langle 2,1,2 \rangle$. Describe the set S.

soln

(a) $\nabla f(P_0) = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle \Big|_{P_0} = \left\langle \frac{1}{y}, -\frac{x}{y^2}, -1 \right\rangle \Big|_{P_0} = \langle 1, -3, -1 \rangle$

(b) Since f is diff'able at P_0 (Check why), for any unit vector $\vec{u}$,

$$D_{\vec{u}} f(P_0) = \vec{u} \cdot \nabla f(P_0)$$

Also recall that the max. rate of change is in the direction of $\nabla f(P_0)$ with magn. $|\nabla f(P)|$

the min " " " " $-\nabla f(P_0)$ " " $-|\nabla f(P)|$

Here, we have

$$-|\nabla f(P_0)| \leq D_{\vec{u}} f(P_0) \leq |\nabla f(P_0)| = \sqrt{11} \quad (< 5)$$

So, $\nexists$ no vector $\vec{u}$ such that $D_{\vec{u}} f(P_0) = 5$

(c) $3 \in [-\sqrt{11}, \sqrt{11}] \Rightarrow$ It is possible to find $\vec{u}$ with $|\vec{u}| = 1$ s.t $D_{\vec{u}} f(P_0) = 3$

i.e $3 = \underbrace{\langle u_1, u_2, u_3 \rangle}_{\vec{u}} \cdot \langle 1, -3, -1 \rangle \iff u_1 - 3u_2 - u_3 = 3$

(with $u_1^2 + u_2^2 + u_3^2 = 1$)

take $\vec{u} = \langle 0, -1, 0 \rangle$

(d) Recall: f increases fastest in the direction of $\nabla f(P)$.

So, the set S is given by

$$S = \left\{ P = (x,y,z) \in \mathrm{Dom}(f) \;\middle|\; \nabla f(P) \parallel \vec{A} = \langle 2,1,2 \rangle \right\}$$

$\hookrightarrow \nabla f(P) = k \cdot \vec{A}$

just rescale $\vec{A}$ by taking $k = -1/2$

$$\langle -1, -\tfrac{1}{2}, \underset{=}{-1} \rangle = \nabla f(P) = \left\langle \frac{1}{y}, -\frac{x}{y^2}, \underset{=}{-1} \right\rangle$$

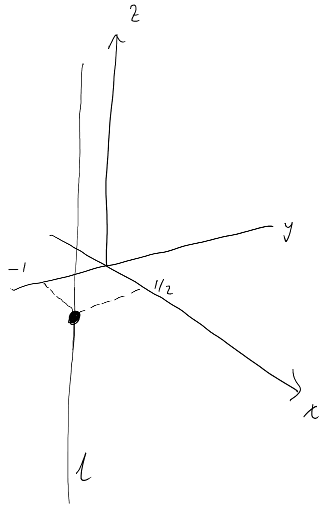
$\iff$

$$\Longleftrightarrow$$

$$\frac{1}{y} = -1 \,, \quad -\frac{x}{y^2} = -1/2$$

$$\Longleftrightarrow$$

$$y = -1 \,, \quad x = \frac{1}{2} \,, \quad z = t \,, \quad t \in \mathbb{R}$$



That means

$$S = \left\{ \left(\tfrac{1}{2}, -1, t \right) : t \in \mathbb{R} \right\}$$

defines a line l with an eqn:

$$\langle x, y, z \rangle = \langle \tfrac{1}{2}, -1, 0 \rangle + t \langle 0, 0, 1 \rangle \,, \quad t \in \mathbb{R} \quad //$$

$\square$

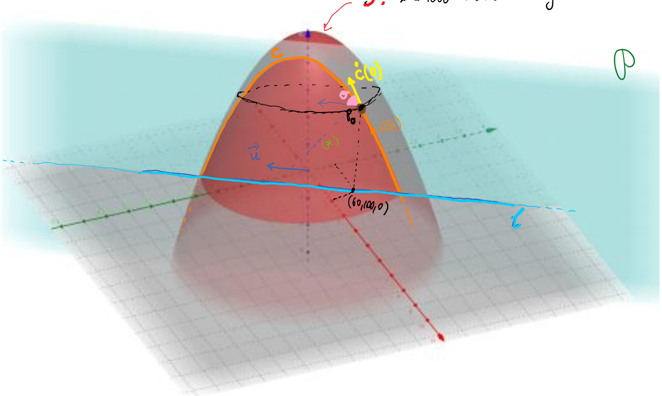
Question-06

Suppose that you are climbing a hill whose shape is given by the equation $z = 1000 - 0.01x^2 - 0.02y^2$ and you are standing at a point with coordinates $(60, 100, 764) = P_0$

(a) In which direction should you proceed initially in order to reach the top of the hill faster?

(b) If you climb in that direction, at what angle above the horizontal will you be climbing initially?

$S: \; z = 1000 - 0.01x^2 - 0.02y^2$



(a) the max. rate of change can be attained in the direction $\nabla f(P_0)$ where

$$\nabla f(P_0) = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle \Big|_{P_0}$$

$$= \langle -1.2, -4 \rangle$$

Denote $\nabla f(P_0)$ by $\vec{u} = \langle -1.2, -4, 0 \rangle$

(b) Harder than it seems ∇ Aim is to determine the angle θ:

- Write down the eqn of the line ℓ passing through the point $(60, 100, 0)$ in the direction of the vector $\vec{u}$:

$$\ell: \quad \langle x, y, z \rangle = \langle 60, 100, 0 \rangle + t \langle -1.2, -4, 0 \rangle$$

or $\quad x = 60 - 1.2t, \quad y = 100 - 4t, \quad$ and $\quad z = 0$

- When z-direction is "free", then the set $\{(x,y,z) \in \mathbb{R}^3 : \; x = 60 - 1.2t, \; y = 100 - 4t\}$ defines a plane $\mathcal{P}$ as in the picture above. Notice that the intersection of the plane $\mathcal{P}$ with the xy-plane is the line ℓ.

- Now, consider the intersection of the surface S and the plane $\mathcal{P}$. Let C be the curve of intersection given by the parametrization

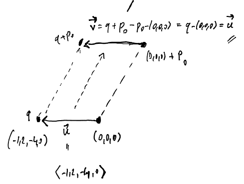
$$c(t) = \langle x(t), y(t), z(t) \rangle = \langle 60 - 1.2t, \; 100 - 4t, \; 1000 - 0.01(60 - 1.2t)^2 - 0.02(100 - 4t)^2 \rangle$$

where $P_0 \in C$ and $P_0 = c(0) = (60, 100, 764)$

Having these observations, [part-b] asks for the angle θ between the vectors $\dot{c}(0)$ and $\vec{u}$

(i.e. we wish to compute the angle between $\dot{c}(0)$ and $\nabla f(P_0)$)

(*) A side remark: While computing, $\vec{u}$ is "adjusted" parallely transported to the vector $\vec{v}$ starting at the point P_0.



Compute: (i) $\dot{c}(0) = \langle -1.2, -4, -0.02(60 - 1.2t)(-1.2) - 0.04(100 - 4t)(-4) \rangle \Big|_{t=0}$

$$= \langle -1.2, -4, 17.44 \rangle$$

and hence $\|\dot{c}(0)\| = \sqrt{18{,}44 \cdot 17{,}44}$

(ii) $\|\vec{u}\| = \|\langle -1.2, -4, 0 \rangle\| = \sqrt{17{,}44}$

Then we have

$$\cos(\theta) = \frac{\dot{c}(0) \cdot \vec{u}}{\|\dot{c}(0)\| \, \|\vec{u}\|} = \frac{\cancel{17{,}44}}{\cancel{17{,}44} \cdot \sqrt{18{,}44}} = \frac{1}{\sqrt{18{,}44}}$$

i.e. $\boxed{\theta = \arccos\left(1 / \sqrt{18{,}44} \right)}$

//

Question-07

Find the equation of the tangent plane to the level surface of the function $f(x, y, z) = xy/z^2$ at the point $P = (1, 2, 3)$.

soln let $t = f(x,y,z) = xy/z^2$.

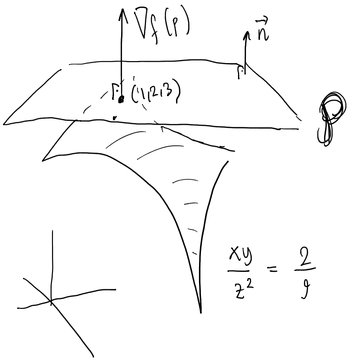
Recall $t = f(x,y,z)$, For each fixed $t = t_0$,

① t can be viewed as a time-parameter.

$t_0 = f(x,y,z)$ gives an instan. picture. (turns out to be a surface)

At $P = (1,2,3)$, the level surface is

$$xy/z^2 = \frac{2}{9} \quad \left(= f(1,2,3) \right)$$



② $\nabla f(p)$ is normal to the level surface of f through the point p.

Compute $\nabla f(p) = \left\langle \frac{y}{z^2}, \frac{x}{z^2}, -\frac{2xy}{z^3} \right\rangle \Big|_{(1,2,3)}$

$$= \left\langle \frac{2}{9}, \frac{1}{9}, -\frac{4}{27} \right\rangle$$

Since $\nabla f(p) \parallel \vec{n}$, just take $\vec{n} = \langle 6, 3, -4 \rangle$.

So, An eqn of the tangent plane ℘ at $(1,2,3)$ is given by

$$6(x-1) + 3(y-2) - 4(z-3) = 0$$

$$\Leftrightarrow$$

$$6x + 3y - 4z = 0$$

Question-08

Check that near the point $(1,0)$ the equation

$$\sin xy + y\ln x + e^{yx} - 1 = 0$$

can be solved for y as a function of x and find the value of $\frac{dy}{dx}$ at the given point.

<u>Soln</u> let $F(x,y) = \sin xy + y\ln x + e^{yx} - 1$ with the eqn $F(x,y) = 0$.

It is enough to check

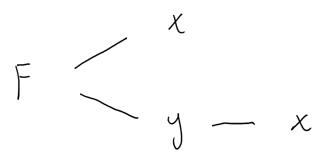
(i) $F_2 \Big|_{(1,0)} \neq 0$ (ii) All partial deriv. exist and are cont. at $(1,0)$.

(clear)

Observe $F_2 \Big|_{(1,0)} = \left(x\cos(xy) + \ln(x) + xe^{yx}\right)\Big|_{(1,0)} = 2 \neq 0$

So, given imp. eqn. can be solved for y as a func. of x, say $y = f(x)$ with $f(1) = 0$

To find $\frac{dy}{dx}$, observe that



Taking deriv. of both sides of $F = 0$ wrt x gives

$$F_1(x,y) + F_2(x,y) \cdot \frac{dy}{dx} = 0$$

So,

$$\frac{dy}{dx} = -\frac{F_1(x,y)}{F_2(x,y)}\Bigg|_{(1,0)} = -\frac{y\cos(xy) + y/x + ye^{xy}}{x\cos(xy) + \ln(x) + xe^{yx}}\Bigg|_{(1,0)} = 0$$