Calculus of Functions of Several Variables
Recitation-01

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Topics to be covered:

Ch. 12: Partial Differentiation **12.1 Functions of Several Variables** **12.2 Limits and Continuity**	**12.1: 4,5,8,12,13,14,20,24** **12.2: 2,6,8,10,12,14,18**
12.3 Partial Derivatives **12.4 Higher-Order Derivatives** **12.5 The Chain Rule**	**12.3: 4,5,6,11,12,16,17,21,24,28,31,36,39** **12.4: 4,10,16** **12.5: 4,8,16,18,29,30**

Question-01

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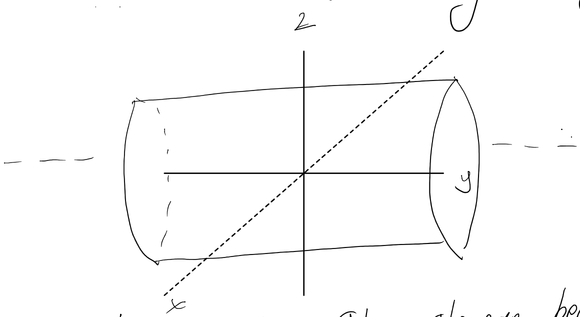
(1) Sketch the graph of the surface $x^2 - cy^2 + z^2 = 1$ for $c \in \mathbb{R}$.

soln Analyze cases:

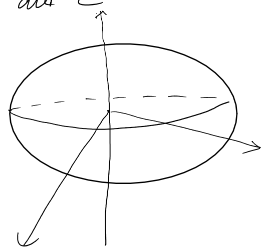
Rmk Forget y-direction, $x^2+z^2=1$ defines a circle of $r=1$ in xz-plane.

- ___ : We have $x^2+z^2=1$

But we have "free" y-direction $\Longrightarrow$ $x^2+z^2=1$ represents a circular cylinder of radius 1 and along the y-axis.



recall $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ defines an ellipsoid with semi-axes a, b and c



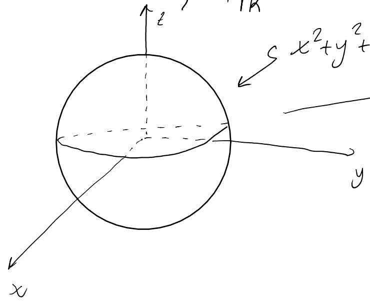
- $c<0$: Note that $-c>0$, let $-c=:K>0$. Then the eqn becomes

$$x^2+Ky^2+z^2=1 \iff \frac{x^2}{1}+\frac{y^2}{(1/\sqrt{K})^2}+\frac{z^2}{1}=1$$

so, it represents an ellipsoid with semi axes 1, $1/\sqrt{K}$ and 1.

When $K=1$, we get a sphere: (unit)

$x^2+y^2+z^2=1$

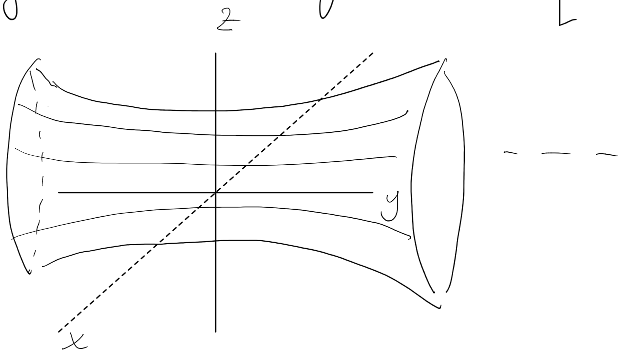


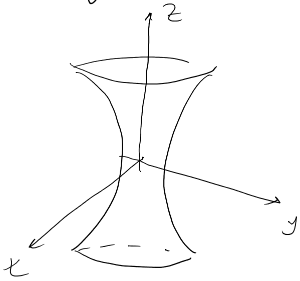
- $c>0$: Given eqn can be rewritten as

$$\rightsquigarrow x^2-cy^2+z^2=1 \iff \frac{x^2}{1^2}+\frac{z^2}{1^2}-\frac{y^2}{(1/\sqrt{c})^2}=1$$

recall $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ reprents an hyperboloid of one sheet.

Be careful! the roles of y and z are interchanged





Question-02

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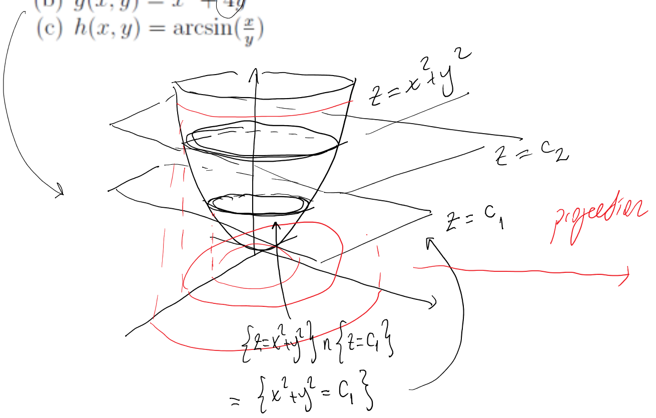
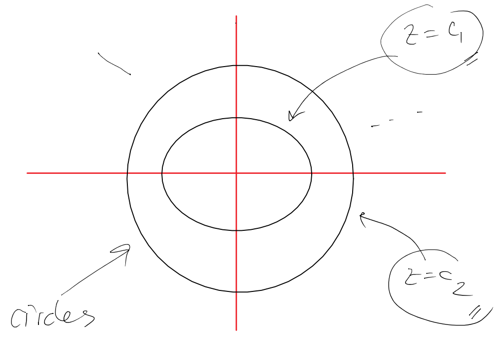
Sketch the level curves and the graphs of the surfaces

(a) $f(x,y) = 2x - y$
(b) $g(x,y) = x^2 + 4y^2$
(c) $h(x,y) = \arcsin(\frac{x}{y})$



Recall

level curves $\equiv$ graph$(f(x,y)) \cap \{$horizontal level planes $z = C\}$



Rmk

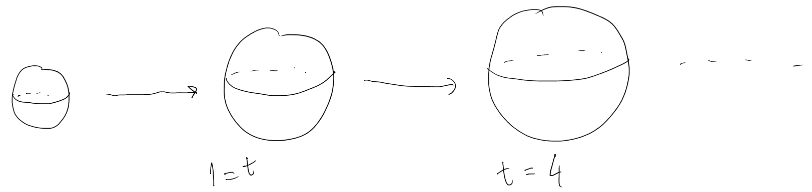
(1) each such $c_i \longleftrightarrow$ "height"

(2) knowing all such curves with their heights $\Longrightarrow$ determine the func.

(3) Similarly, idea works in higher dims., e.g. 4D.

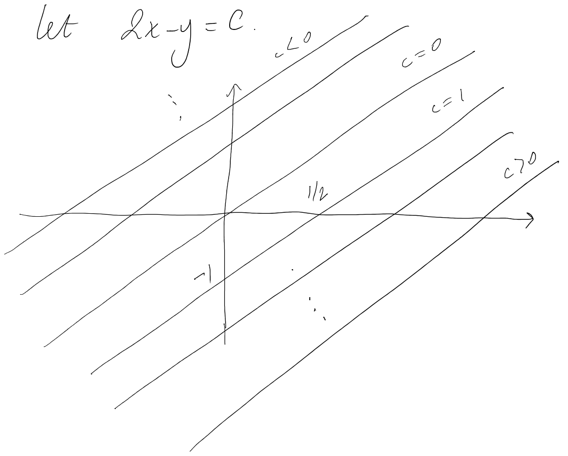
$\hookrightarrow$ consider $t = f(x,y,z)$, For fixed $t = C$, $C = f(x,y,z)$ gives "level surface"

e.g. $t = x^2+y^2+z^2$ $\quad 0{,}5 = x^2+y^2+z^2$

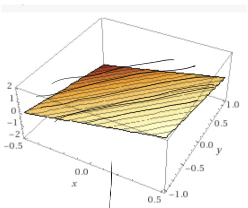


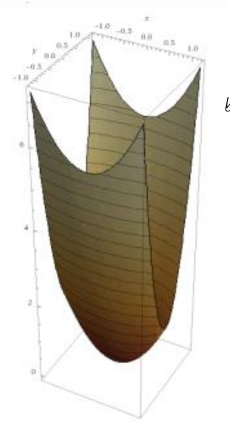
(a) Given $z = 2x - y$, let $2x - y = c$.

plane in $\mathbb{R}^3$



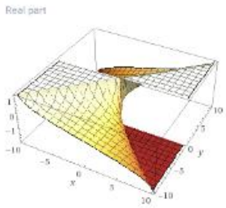
$z = 2x - y$

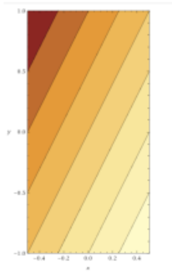


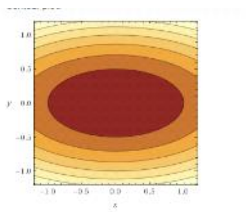


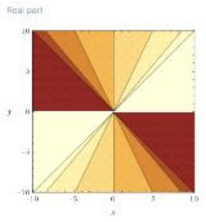
$z = x^2 + 4y^2$

$z = \arcsin\left(\frac{x}{y}\right)$









Question-03

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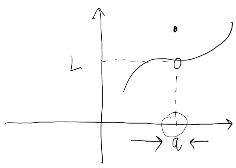
Evaluate the indicated limits or explain why they do not exist

(a) $\lim\limits_{(x,y)\to(0,0)} \cancel{\dfrac{xy+y^4}{x^2+y^4}}$ (change the function) $\dfrac{2x^2+y^4}{x^2+3y^4}$

(b) $\lim\limits_{(x,y)\to(1,2)} \dfrac{2x^2-xy}{4x^2-y^2}$

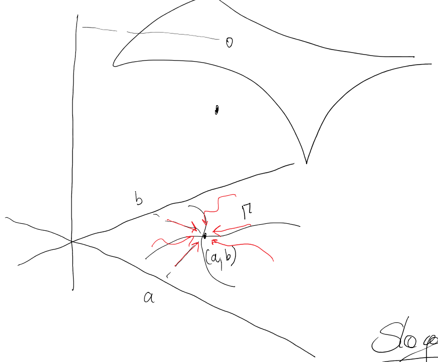
(d) $\lim\limits_{(x,y)\to(0,0)} \dfrac{(\sin^2 x)(\sin^2 y)}{(x^2+y^2)^2}$

Recall One-variable limits



$$\boxed{\lim_{x\to a} f(x) \text{ exists (and } = L) \iff \lim_{x\to a^+} f(x) = L = \lim_{x\to a^-} f(x)}$$

In higher dim's;



So, If $\lim\limits_{(x,y)\to(a,b)} f(x,y) = L$ exists, then

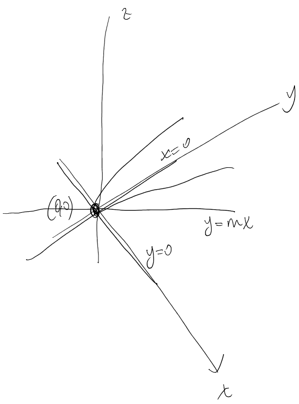
$\lim\limits_{\substack{(x,y)\to(a,b)\\ \text{on } \Gamma}} f$ exists and $\lim\limits_{\substack{(x,y)\to(a,b)\\ \text{on } \Gamma}} f = L$ $\forall$ curves Γ passing through (a,b)

Slogan: showing non-existence is easier!

(a) Check the limit on some curves: $\Gamma_1 : x=0$,

$\Gamma_2 : y=0$,

$\Gamma_3 : y=mx$, or $\Gamma_4 : y = mx^2$ ....



on $\Gamma_1 : x=0$; $\lim\limits_{(x,y)\to(0,0)} \dfrac{2x^2+y^4}{x^2+3y^4} \underset{\text{on }\Gamma_1}{=} \lim\limits_{y\to 0} \dfrac{\cancel{y^4}}{3\cancel{y^4}} = \dfrac{1}{3}$

$\neq$

on $\Gamma_2 : y=0$; $\lim\limits_{(x,y)\to(0,0)} \dfrac{2x^2+y^4}{x^2+3y^4} \underset{\text{on }\Gamma_2}{=} \lim\limits_{x\to 0} \dfrac{2\cancel{x^2}}{\cancel{x^2}} = 2$

Since $\lim\limits_{\substack{(x,y)\to(0,0)\\ \text{on }\Gamma_1}} f(x,y) \neq \lim\limits_{\substack{(x,y)\to(0,0)\\ \text{on }\Gamma_2}} f(x,y)$, $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ d.n.e.

(b) $\lim\limits_{(x,y)\to(1,2)} \dfrac{2x^2-xy}{4x^2-y^2} = \lim\limits_{(x,y)\to(1,2)} \dfrac{x\cancel{(2x-y)}}{\cancel{(2x-y)}(2x+y)} = \dfrac{1}{4}$.

(c) Observe that

on $\Gamma_1 : x=0$, we have

$$\lim_{\substack{(x,y)\to(0,0)\\ \text{on }\Gamma_1}} f(x,y) = \lim_{y\to 0} \frac{0}{y^4} = 0.$$

on $\Gamma_2 : y=x$, we have

$\lim \dfrac{\sin^2 x \sin^2 y}{\dots} = \lim \dfrac{\sin^2 x \sin^2 x}{\dots}$

on $\Gamma_2 : y = x$, we have

$$\lim_{(x,y)\to(0,0)} \frac{\sin^2 x \sin^2 y}{(x^2+y^2)^2} \underset{\text{on } \Gamma_2}{=} \lim_{x\to 0} \frac{\sin^2 x \sin^2 x}{(2x^2)^2}$$

$$= \lim_{x\to 0} \left(\frac{\sin x}{x}\right)^4 \cdot \frac{1}{4} \qquad \underline{\text{recall}} \ \lim_{t\to 0} \frac{\sin t}{t} = 1$$

$$= 1/4$$

Here $\displaystyle\lim_{\substack{(x,y)\to(0,0)\\ \text{on } \Gamma_1}} f(x,y) \neq \lim_{\substack{(x,y)\to(0,0)\\ \text{on } \Gamma_2}} f(x,y)$. So, $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y)$ $\underline{\text{d.n.e.}}$

$\square$

Question-04

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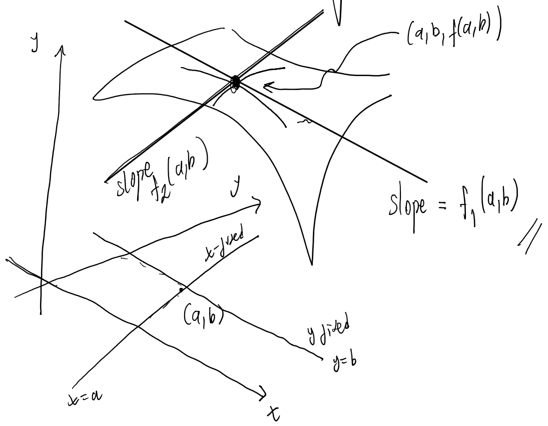
Determine $f_1(0,0)$ and $f_2(0,0)$ (if they exists) where

$$f(x,y) = \begin{cases} (x^3+y)\sin\left(\frac{1}{x^2+y^2}\right) & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

soln

Note that the beh. of the func. changes around the pt $(0,0)$, we should use the defns !



recall

$$f_1(a,b) = \lim_{h\to 0} \frac{f(a+h,\, b) - f(a,b)}{h}$$

$$f_2(a,b) = \lim_{h\to 0} \frac{f(a,\, b+h) - f(a,b)}{h}$$

In our case,

(i)
$$f_1(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{h^3 \sin\frac{1}{h^2} - 0}{h}$$

$$= \lim_{h\to 0} \frac{\sin(1/h^2)}{1/h^2}, \quad \text{let } \frac{1}{h^2} =: t$$

$$= \lim_{t\to\infty} \frac{\sin(t)}{t} \quad \text{where } -\frac{1}{t} \leq \frac{\sin(t)}{t} \leq \frac{1}{t} \quad (*)$$

$$= 0 \quad \text{by Squeeze thm with } (*)$$

(ii)
$$f_2(0,0) = \lim_{h\to 0} \frac{f(0,h) - f(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{h}\sin(1/h^2)}{\cancel{h}} \quad \text{does not exist.}$$

Question-05

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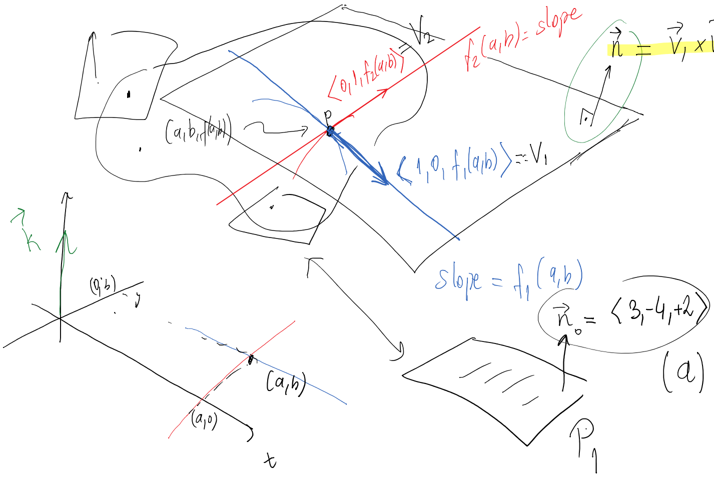
Find all points on the surface $z = \underbrace{x^2 - 2xy - y^2 - 8x + 4y}_{f(x,y)}$ where the tangent plane is (a) horizontal (b) parallel to the plane $3x - 4y + 2z = 7$.



$\vec{n} = \vec{V}_1 \times \vec{V}_2 = \langle -f_1(a,b), -f_2(a,b), 1 \rangle$

an ~~eqn~~ eqn of the tangent plane to $z = f(x,y)$ at $(a, b, f(a,b))$ is

$$\boxed{z = f(a,b) + f_1(a,b)(x-a) + f_2(a,b)(y-b)}$$

(a) look at points (x,y) s.t $\vec{n} \parallel \vec{k} = \langle 0, 0, 1 \rangle$

i.e. find $(x,y) \in \mathbb{R}^2$ such that

$$f_1(x,y) = 0 = f_2(x,y)$$

Observe that let $f(x,y) = x^2 - 2xy - y^2 - 8x + 4y$,

$$f_1(x,y) = 2x - 2y - 8$$
$$f_2(x,y) = -2x - 2y + 4$$

$$\left.\begin{aligned} 2x - 2y - 8 &= 0 \\ -2x - 2y + 4 &= 0 \end{aligned}\right\} \Rightarrow \boxed{(x,y) = (3,-1)}$$

$$x - y = 4$$
$$-x - y = -2$$
$$-2y = 2 \Rightarrow \boxed{y = -1}, \boxed{x = 3}$$

(b) note that the tangent plane is $\parallel P_1$ iff $\vec{n} \parallel \langle 3, -4, 2 \rangle$

i.e $\vec{n} = k \langle 3, -4, 2 \rangle$, $k \in \mathbb{R}$.

For simplicity, look for points $(x,y) \in \mathbb{R}^2$ s.t.

$$f_1(x,y) = -\frac{3}{2} \quad \text{and} \quad f_2(x,y) = 2$$

solving the liner system $\left.\begin{aligned} 2x - 2y - 8 &= -3/2 \\ -2x - 2y + 4 &= 2 \end{aligned}\right\}$ gives $\boxed{(x,y) = \left(\frac{17}{8}, -\frac{9}{8}\right)}$

Question-06

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Find all the second order partial derivatives of the given function:

(a) $z = xe^y - ye^x$

(b) $f(x,y) = \ln(\underbrace{1+\cos(xy)}_{t})$

(a) Let $f(x,y) = xe^y - ye^x$. Then

$f_x = e^y - ye^x$, $\boxed{f_{xy} = e^y - e^x}$, $f_{xx} = 0 - ye^x$

(or f_1) (f_{12}) (f_{11})

$f_y = xe^y - e^x$, $\boxed{f_{yx} = e^y - e^x}$, $f_{yy} = xe^y$

(b) $f_x = \dfrac{-1}{1+\cos(xy)} \cdot \sin(xy) \cdot y$

$f_{xy} = \left[\dfrac{-1}{(1+\cos(xy))^2} \cdot \sin(xy) \cdot x\right] (\sin(xy)\, y) + \left(\dfrac{-1}{1+\cos(xy)}\right) \left[\cos(xy)\, x \cdot y + \sin(xy) \cdot 1\right]$

$f_{yx} = \ldots$

$f_{xx} = \ldots$

exc. Compute the others...

Question-07

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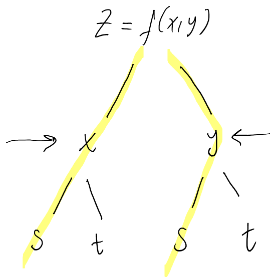
Suppose $z = f(x, y)$ is a function with continuous second order partial derivatives. Suppose that $x = e^s t$, $y = se^t$ and

$f(0, 1) = -2$, $f_x(0, 1) = 2$, $f_y(0, 1) = 4$, $f_{xx}(0, 1) = 3$, $f_{xy}(0, 1) = 1$, $f_{yy}(0, 1) = -1$

$f(1, 0) = -1$, $f_x(1, 0) = -1$, $f_y(1, 0) = 2$, $f_{xx}(1, 0) = -1$, $f_{xy}(1, 0) = 1$, $f_{yy}(1, 0) = 3$

(a) Compute $\frac{\partial f}{\partial s}$ and $\frac{\partial f}{\partial t}$ at the point $(s, t) = (1, 0)$.
(b) Compute $\frac{\partial^2 f}{\partial s \partial t}$ at the point $(s, t) = (1, 0)$



$x = e^s t$
$y = se^t$

(a) Use the chain rule wrt "dependence relation / the chain":

Compute:

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}$$

$$= f_x e^s t + f_y e^t$$

so,

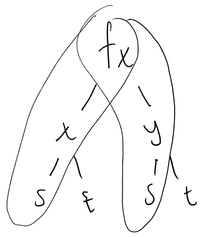
$$\left.\frac{\partial f}{\partial s}\right|_{\substack{(s,t)=(1,0)\\ ((x,y)=(0,1))}} = \cancel{f_x(0,1)\cdot e^1 \cdot 0} + f_y(0,1)\cdot e^0 = 4$$

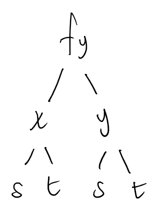
0

exc: Compute $\left.\frac{\partial f}{\partial t}\right|_{(s,t)=(1,0)}$

(b) Compute $\frac{\partial}{\partial s}\left(\frac{\partial f}{\partial t}\right)$: where $\boxed{\frac{\partial f}{\partial t} = f_x e^s + f_y s e^t}$ $(*)$

Rmk Apply chain rule for f_x and f_y wrt the dependence relation





Then we have

$$\frac{\partial}{\partial s}\left(\frac{\partial f}{\partial t}\right) = \underbrace{\frac{\partial f_x}{\partial s}}\cdot e^s + f_x e^s + \underbrace{\frac{\partial f_y}{\partial s}}\cdot se^t + f_y \cdot e^t$$

$$= \left[f_{xx}\frac{\partial x}{\partial s} + f_{xy}\frac{\partial y}{\partial s}\right]e^s + f_x \cdot e^s + \left[f_{yx}\frac{\partial x}{\partial s} + f_{yy}\frac{\partial y}{\partial s}\right]e^t s + f_y e^t$$

and then we get

compute using the data given

$$\left.\frac{\partial^2 f}{\partial s \partial t}\right|_{(s,t)=(1,0)} = 3e + 3$$

$'(s,t) = (1,0)$

$(x,y) = (0,1)$

$\square$