Math 118, Chapter 9.7, Recitation Notes

Q1) Find the Maclaurin series of following functions

a) $\int_0^x t^2 \arctan(t^4)\,dt$ b) $\int_0^x \left(\frac{e^t-1}{t}\right)dt$

Solution)

a) Recall: (Geometric Series)

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \Longrightarrow \frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

$$\arctan x = \int \frac{dx}{1+x^2} \Longrightarrow \arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)}$$

$$\Longrightarrow \arctan(t^4) = \sum_{n=0}^{\infty} \frac{(-1)^n t^{8n+4}}{(2n+1)}$$

$$\Longrightarrow t^2 \arctan(t^4) = \sum_{n=0}^{\infty} \frac{(-1)^n t^{8n+6}}{(2n+1)}$$

$$\int_0^x t^2 \arctan(t^4)\,dt = \int_0^x \left(\sum_{n=0}^{\infty} \frac{(-1)^n t^{8n+6}}{(2n+1)} \right) dt$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \int_0^x t^{8n+6}\,dt = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \left[\frac{t^{8n+7}}{8n+7} \right]_0^x$$

$$\Longrightarrow \int_0^x t^2 \arctan(t^4)\,dt = \sum_{n=0}^{\infty} \frac{(-1)^n x^{8n+7}}{(2n+1)(8n+7)}$$

b) $\int_0^x \left(\frac{e^t-1}{t}\right) dt$

Recall: $e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!}$

$\Rightarrow e^t - 1 = \sum_{n=1}^{\infty} \frac{t^n}{n!} \Rightarrow \frac{e^t-1}{t} = \sum_{n=1}^{\infty} \frac{t^{n-1}}{n!}$

$$\Rightarrow \int_0^x \left(\frac{e^t-1}{t}\right) dt = \int_0^x \left[\sum_{n=1}^{\infty} \frac{t^{n-1}}{n!}\right] dt$$

$$= \sum_{n=1}^{\infty} \int_0^x \frac{t^{n-1}}{n!} dt = \sum_{n=1}^{\infty} \left(\frac{t^n}{n.n!}\right)\Big|_0^x = \sum_{n=1}^{\infty} \frac{x^n}{n.n!}$$

$$\Rightarrow \int_0^x \left(\frac{e^t-1}{t}\right) dt = \sum_{n=1}^{\infty} \frac{x^n}{n.n!}$$

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Q2 Estimate the following numbers with error less than 0.0005.

a) $\sin\left(\frac{\pi}{10}\right)$ b) $\ln(1.01)$

Solution: We will use "error for alternating series".

Recall: (Alternating Series Test)

Suppose that $\sum_{k=0}^{\infty} a_k$ satisfies 1,2,3.

Then the series is convergent, where...

1) $a_k \cdot a_{k+1} < 0$

2) $|a_k| > |a_{k+1}|$

3) $\lim_{k\to\infty} |a_k| = 0$

Recall: (Error for Alternating Series)

Suppose that $\sum_{k=0}^{\infty} a_k$ satisfies AST.

$\Rightarrow$ Then we know that it converges.

Set $S = \sum_{k=0}^{\infty} a_k$. Then we have the following error formula.

$$E_J := \left|S - \sum_{k=0}^{J} a_k\right| < |a_{J+1}|$$

Now, for the solution we will approximate given numbers using alternating series.

a) $\sin\left(\frac{\pi}{10}\right)$

Recall: $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \quad \forall x \in \mathbb{R}$

$$\Rightarrow \sin\left(\frac{\pi}{10}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{\pi}{10}\right)^{2n+1}}{(2n+1)!}$$

Set $\quad a_n := \frac{(-1)^n \left(\frac{\pi}{10}\right)^{2n+1}}{(2n+1)!}$

Exercise: Check $\sum_{n=0}^{\infty} a_n$ satisfies conditions 1, 2, 3. for Alt. Series Test

$\Rightarrow$ By exercise, we can use error for Alternating series.

$$|a_3| = \left| \frac{(-1)^3 \left(\frac{\pi}{10}\right)^7}{7!} \right| \leq \left(\frac{1}{2}\right)^7 \cdot \frac{1}{7!} < 0.0005$$

Then by error for alternating series

$$\left| \sin\left(\frac{\pi}{10}\right) - \left[\frac{\pi}{10} - \frac{\left(\frac{\pi}{10}\right)^3}{3!} + \frac{\left(\frac{\pi}{10}\right)^5}{5!}\right] \right| < |a_3| < 0.0005$$

$$\sin\left(\frac{\pi}{10}\right) \cong \frac{\pi}{10} - \frac{\left(\frac{\pi}{10}\right)^3}{3!} + \frac{\left(\frac{\pi}{10}\right)^5}{5!}$$

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b) $\ln(1.01)$

Recall: $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad |x|<1.$

$\Rightarrow \frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n \quad |x|<1.$

$\ln(1+x) = \int \frac{dx}{1+x} \Rightarrow \ln(1+x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1}$

$|x|<1.$ $\circledast$

Since $|0.01|<1$ we can use $\circledast$ to obtain a formula for $\ln(1.01)$

$$\ln(1.01) = \ln(1+0.01) = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{1}{100}\right)^{n+1}}{n+1}$$

Set $a_n = \frac{(-1)^n \left(\frac{1}{100}\right)^{n+1}}{n+1}$.

Exercise: Show $\sum_{n=0}^{\infty} a_n$ satisfies 1,2,3 for AST

Then by exercise we can use error for AST.

$$|a_1| = \left| \frac{(-1)\left(\frac{1}{100}\right)^2}{2} \right| < 0.0005$$

$$\left| \ln(1.01) - a_0 \right| = \left| \ln(1.01) - \frac{1}{100} \right| < |a_1| < 0.0005$$

$$\Rightarrow \ln(1.01) \sim \frac{1}{100}$$

(Q3) Compute the following limit.

$$\lim_{x \to 0} \frac{\sin 3x \,(1-\cos 2x)}{e^{x^3}-1}$$

Solution:

$$\sin(u) = \sum_{n=0}^{\infty} \frac{(-1)^n u^{2n+1}}{(2n+1)!}$$

$$\Rightarrow \sin 3x = \sum_{n=0}^{\infty} \frac{(-1)^n 3^{2n+1} x^{2n+1}}{(2n+1)!}$$

$$\cos u = \sum_{n=0}^{\infty} \frac{(-1)^n u^{2n}}{(2n)!}$$

$$\Rightarrow 1-\cos 2x = 1 - \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n} x^{2n}}{(2n)!}$$

$$= -\sum_{n=1}^{\infty} \frac{(-1)^n 4^n x^{2n}}{(2n)!}$$

Recall: (Cauchy Product)

$$f(x) = \sum a_n x^n \quad |x| < r_1$$

$$g(x) = \sum b_k x^k \quad |x| < r_2$$

$$\Rightarrow f(x)g(x) = \sum_j c_j x^j \quad |x| < \min\{r_1, r_2\}$$

$$c_j = \sum_{n=0}^{j} a_n b_{j-n}$$

(We can multiply power series like polynomials)

$\Rightarrow$ Cauchy Product gives

$$\sin 3x(1-\cos 2x) = \left[\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} 3^{2n+1}\right]\left[\sum_{j=1}^{\infty} \frac{(-1)^j 4^j x^{2j}}{(2j)!}\right]$$

$$= \left[3x - \frac{9}{2}x^3 + \frac{3^5}{5!}x^5 - \dots\right]\left[2x^2 - \frac{2}{3}x^4 - \dots\right]$$

$$= \left[6x^3 - 11x^5 - \dots\right]$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \Rightarrow e^{x^3} - 1 = \sum_{n=1}^{\infty} \frac{x^{3n}}{n!} = \left[x^3 + \frac{x^6}{2} + \frac{x^9}{6} - \dots\right]$$

$\Rightarrow$ We will substitude Maclaurin series for numerator and denominator.

$$\lim_{x\to 0} \frac{\sin 3x(1-\cos 2x)}{e^{x^3}-1} = \lim_{x\to 0} \frac{6x^3 - 11x^5 - \dots}{x^3 + \frac{x^6}{2} + \frac{x^9}{6} - \dots}$$

$$= \lim_{x\to 0} \frac{6 - 11x^2 - \dots}{1 + \frac{x^3}{2} + \frac{x^6}{6} - \dots} = \frac{6}{1} = 6$$

$$\therefore \lim_{x\to 0} \frac{\sin 3x(1-\cos 2x)}{e^{x^3}-1} = 6$$

Q4) $f(x) = \dfrac{\cos x - 1 + \frac{x^2}{2} - \frac{x^4}{24}}{x^6} \quad x \neq 0$

Given that $f(0)$ is choosen so that f is continious at $x = 0$ (in this case f will have a power series expansion about $x=0$) find a) $f(0)$, b) $f^{(4)}(0)$

Solution:

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \Longrightarrow$$

$$\frac{\cos x - 1 + \frac{x^2}{2} - \frac{x^4}{24}}{x^6} = \sum_{n=3}^{\infty} \frac{(-1)^n x^{2n-6}}{(2n)!}$$

f is continious at $x = 0$

$$\boxed{\begin{aligned} f(0) &= \lim_{x \to 0} f(x) = \lim_{x \to 0} \left[\frac{\cos x - 1 + \frac{x^2}{2} - \frac{x^4}{24}}{x^6} \right] \\ &= \lim_{x \to 0} \sum_{n=3}^{\infty} \frac{(-1)^n x^{2n-6}}{(2n)!} = \frac{-1}{6!} = \frac{-1}{120} \end{aligned}}$$

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Recall: (Taylor's theorem)

If f has a Maclaurin series expansion about $x=0$ then it is given as follows.

$$f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n$$

$$\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n = f(x) = \frac{\cos x - 1 + \frac{x^2}{2} - \frac{x^4}{24}}{x^6} = \sum_{j=3}^{\infty}\frac{(-1)^j x^{2j-6}}{(2j)!}$$

Two power series are equal if and only if coefficients of powers of x are equal on both series.

Hence coefficients of x^4 are equal on both series.

$$\frac{f^{(4)}(0)}{4!} = \frac{(-1)^5}{(10)!}$$

$$\Rightarrow f^{(4)}(0) = \frac{-1}{(10)!/4!} = \frac{(-1)}{10.9.8.7.6.5}$$