Math 118 Chapter 9.6 (Power Series)

Note Title

Q1: Let $f(x) = \sum_{n=0}^{\infty} \frac{1}{n+2} \left(\frac{x-2}{3} \right)^{2n+1}$ for $-1 \le x < 5$. Find $f^{(50)}(2)$ and $f^{(51)}(2)$.

$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n = f(c) + \frac{f'(c)}{1!}(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots$

$f^{(n)}(c)$ appears in the coefficient of $(x-c)^n$ in the above ~~expansion~~ expansion.

Soln:

$f^{(50)}(2) = ?$ appears in the coef of $(x-2)^{50}$.

$2n+1 = 50 \Rightarrow$ has no integer soln.

So, $\frac{f^{(50)}(2)}{50!} = 0 \Rightarrow \boxed{f^{(50)}(2) = 0}$

$f^{(51)}(2) = ?$ appears in the coef. of $(x-2)^{51}$.

$2n+1 = 51 \Rightarrow n = 25$

$\frac{f^{(51)}(2)}{51!} = \frac{1}{27} \frac{1}{3^{51}} \Rightarrow \boxed{f^{(51)}(2) = \frac{51!}{27 \cdot 3^{51}}}$ //

Q2 : Find the Taylor series representations of the following functions

- e^{-2x} about $c=-1$

$$f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(-1)}{n!}(x+1)^n$$

$f^{(0)}(x)=f(x)=e^{-2x}\longrightarrow f(-1)=e^{2}$

$f'(x)=-2e^{-2x}\longrightarrow f'(-1)=-2e^{2}$

$f''(x)=2^2e^{-2x}\longrightarrow f''(-1)=2^2e^{2}$

$f'''(x)=-2^3e^{-2x}\longrightarrow$

$\vdots$

- $f^{(n)}(x)=(-1)^n2^ne^{-2x}\longrightarrow f^{(n)}(-1)=(-1)^n2^ne^{2}$

Verification : Induction.

$n=0$, $f^{(0)}(x)=(-1)^0 2^0 e^{-2x}=e^{-2x}=f(x)$ ✓

Say the formula is true for n.

$\frac{d}{dx}\left[(-1)^n2^ne^{-2x}\right]=(-1)^{n+1}2^{n+1}e^{-2x}=f^{(n+1)}(x)$ ✓

So,

$$f(x)=\sum_{n=0}^{\infty}\frac{(-1)^n2^ne^{2}}{n!}(x+1)^n$$

Interval of convergence

$$\lim_{n\to\infty}\left|\frac{(-1)^{n+1}2^{n+1}e^{2}(x+1)^{n+1}}{(n+1)!}\cdot\frac{n!}{(-1)^n2^ne^{2}(x+1)^n}\right|$$

$$=\lim_{n\to\infty}\left|\frac{2(x+1)}{n+1}\right|=0<1.$$

So, interval of convergence is $(-\infty,\infty)$ $\mathbb{R}$.

• $\cos^2\left(\frac{x}{2}\right)$ about 0

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

$f(x) = \cos^2\left(\frac{x}{2}\right) \longrightarrow f(0) = 1.$

$f'(x) = -2\cos\left(\frac{x}{2}\right)\sin\left(\frac{x}{2}\right)\cdot\frac{1}{2} = \frac{-\sin x}{2}. \longrightarrow f'(0) = 0$

$f''(x) = \frac{-\cos x}{2} \longrightarrow f''(0) = -\frac{1}{2}$

$f'''(x) = \frac{\sin x}{2} \longrightarrow f'''(0) = 0.$

$f^{IV}(x) = \frac{\cos x}{2} \longrightarrow f^{IV}(0) = \frac{1}{2}$

$f^{VI}(x) = \frac{-\sin x}{2} \longrightarrow f^{V}(0) = 0.$

So,

$$f(x) = 1 + \frac{0}{1!}x + \frac{-1}{2\cdot 2!}x^2 + \frac{0}{3!}x^3 \cdots$$

$$f^{(2n)}(0) = \frac{(-1)^n}{2}$$

Soln # 2: $\cos^2\frac{x}{2} = \frac{1}{2}(\cos x + 1) = \frac{1}{2}\cos x + \frac{1}{2}$

Also, $\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \quad x \in \mathbb{R}.$

$$\cos^2\frac{x}{2} = \frac{1}{2} + \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{2(2n)!}$$

for all $x \in \mathbb{R}$.

• $\sin(x)$ about $\frac{\pi}{2}$ $\quad c=\frac{\pi}{2}$

$f(x) = \sin x \longrightarrow f\left(\frac{\pi}{2}\right) = 1$

$f'(x) = \cos x \qquad f'\left(\frac{\pi}{2}\right) = 0$

$f''(x) = -\sin x \qquad f''\left(\frac{\pi}{2}\right) = -1$

$f'''(x) = -\cos x \qquad f'''\left(\frac{\pi}{2}\right) = 0$

$f^{IV}(x) = \sin x \qquad f^{IV}\left(\frac{\pi}{2}\right) = 1.$

So, $f^{(2k)}\left(\frac{\pi}{2}\right) = (-1)^k.$

So,

$$\sin x = \sum_{k=0}^{\infty} \frac{f^{(2k)}(\pi/2)}{(2k)!}\left(x-\frac{\pi}{2}\right)^{2k} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}\left(x-\frac{\pi}{2}\right)^{2k}$$

Radius of convergence.

$$\lim_{k\to\infty} \left| \frac{(-1)^{k+1}}{(2k+2)!}\left(x-\frac{\pi}{2}\right)^{2k+2} \frac{(2k)!}{(-1)^k\left(x-\frac{\pi}{2}\right)^{2k}} \right| = \lim_{k\to\infty} \left| \frac{\left(x-\frac{\pi}{2}\right)^2}{(2k+1)(2k+2)} \right| = 0 < 1$$

So, radius of convergence is infinity.

Interval of conv. is $\mathbb{R}$.

- $\ln(2+x^2)$ about 0

$f(x) = \ln(2+x^2)$

$f'(x) = \dfrac{2x}{2+x^2}$

$f''(x) = \dfrac{-2(x^2-2)}{(2+x^2)^2}$

$f'''(x) = \dfrac{4x(x^2-6)}{(2+x^2)^3}$

$\vdots$

no general pattern.

Soln #1:

Consider $f'(x) = \dfrac{2x}{2+x^2} = \dfrac{2x}{2\left(1+\frac{x^2}{2}\right)}$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad |x|<1$$

$$\frac{1}{1+\frac{x^2}{2}} = \frac{1}{1-\left(-\frac{x^2}{2}\right)} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{2^n}, \quad \left|-\frac{x^2}{2}\right| = \left|\frac{x^2}{2}\right| < 1$$

So, $\dfrac{x}{1+\frac{x^2}{2}} = \displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2^n}$ whenever $|x|<\sqrt{2}$

Thus,

$$\ln(2+x^2) = \int \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2^n}\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n} \int x^{2n+1}\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{2^n(2n+2)} + C$$

$\ln 2$, 0, $C = \ln 2$

$$\text{So, } \ln(2+x^2) = \ln 2 + \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{2^n(2n+2)} = \ln 2 + \frac{x^2}{2} - \frac{x^4}{8} + \frac{x^6}{4\cdot 6} - \cdots \quad |x|<\sqrt{2}$$

Soln # 2: $\ln(2+x^2)$?

$\ln(2+x) = \ln\left(2\left(1+\frac{x}{2}\right)\right) = \ln 2 + \ln\left(1+\frac{x}{2}\right)$.

Recall from the prev recitation:

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$$

$$\ln\left(1+\frac{x}{2}\right) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n2^n}$$

$$\ln(2+x) = \ln 2 + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n2^n}$$

$$\ln(2+x^2) = \ln 2 + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{2n}}{n2^n} = \ln 2 + \frac{x^2}{2} - \frac{x^4}{8} + \cdots$$

Q3: Find the sum of the following series

$$x^3 - \frac{x^9}{3!\cdot 4} + \frac{x^{15}}{5!\cdot 16} - \frac{x^{21}}{7!\cdot 64} + \cdots$$

2^2 $\quad$ 2^4 $\quad$ 2^6

Soln:

$$= 2\left[\frac{x^3}{2} - \frac{x^9}{3!\cdot 2^3} + \frac{x^{15}}{5!\cdot 2^5} - \frac{x^{21}}{7!\cdot 2^7} + \cdots\right]$$

$$= 2\left[\frac{x^3}{2} - \frac{1}{3!}\left(\frac{x^3}{2}\right)^3 + \frac{1}{5!}\left(\frac{x^3}{2}\right)^5 - \frac{1}{7!}\left(\frac{x^3}{2}\right)^7 + \cdots\right]$$

$$= 2\sin\frac{x^3}{2} \quad \checkmark$$

$x \in \mathbb{R}$

Recall: $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$

$$1+\frac{1}{2\cdot 2!}+\frac{1}{4\cdot 3!}+\frac{1}{8\cdot 4!}+\cdots$$

$$e^x = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\cdots \quad x\in\mathbb{R}$$

$$e^{1/2} = 1+\underbrace{\frac{1}{2}+\frac{1}{2!}\left(\frac{1}{2}\right)^2+\frac{1}{3!}\cdot\left(\frac{1}{2}\right)^3+\cdots}$$

Soln:

$$= 2\left[\underbrace{\frac{1}{2}+\frac{1}{2!}\cdot\left(\frac{1}{2}\right)^2+\frac{1}{3!}\cdot\left(\frac{1}{2}\right)^3+\cdots}\right]$$

$$= 2\left[e^{1/2}-1\right]$$

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Q4: Write the first three nonzero terms of the Maclaurin series for $\sin^{-1}(x)$.

$c=0$

Soln:

$f(x) = \sin^{-1} x$ — — — — $f(0)=0$

$f'(x) = (1-x^2)^{-1/2}$ $\quad f'(0)=1$

$f''(x) = -\frac{1}{2}(1-x^2)^{-3/2}(-2x) = x(1-x^2)^{-3/2}$ $\quad f''(0)=0$

$f'''(x) = (1-x^2)^{-5/2}(2x^2+1)$ HW $\quad f'''(0)=1$

$f^{IV}(x) = 3x(1-x^2)^{-7/2}(2x^2+3)$ $\quad f^{IV}(0)=0$

$f^{V}(x) = 3(1-x^2)^{-9/2}(3+24x^2+8x^4)$. $\quad f^{V}(0)=9$.

$x + \frac{1}{3!}x^3 + \frac{9}{5!}x^5 + \cdots$

Q5: If $f(x) = \tan^{-1}(x)$ find $f^{(99)}(0) = ?$

Soln: $\tan^{-1}x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \longrightarrow$ if $2n+1 = 99 \Rightarrow n = 49$

$$\frac{(-1)^{49}}{99}$$

$$a_{99} = \frac{f^{(99)}(0)}{99!} \Rightarrow f^{(99)}(0) = 99! \cdot a_{99} = 99! \cdot \frac{(-1)^{49}}{99} = -98!$$

the coef. of x^{99} in the Taylor series expansion of $\tan^{-1}x$.