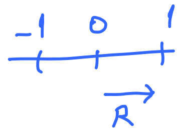
Math 118 – Chapter 9.5 (Power Series)

Note Title

Q1: Determine the center, radius and the interval of convergence of the following series.

1a. $\displaystyle\sum_{n=1}^{\infty} \frac{x^{2n}}{\sqrt{2n+1}}$ $\quad c=0$



Soln: $\displaystyle\lim_{n\to\infty}\left|\frac{x^{2n+2}}{\sqrt{2n+3}}\cdot\frac{\sqrt{2n+1}}{x^{2n}}\right| = \lim_{n\to\infty}\left|x^2\frac{\sqrt{2n+1}}{\sqrt{2n+3}}\right| = |x^2| = x^2 < 1 \Rightarrow$ $\therefore x\in(-1,1)$.

Radius, $R=1$

Endpoints

If $x=1$ then $\displaystyle\sum_{n=1}^{\infty}\frac{1}{\sqrt{2n+1}}$ diverges

[comparison, or, LC, $\displaystyle\sum_{n=1}\frac{1}{\sqrt{n}}$; $\sqrt{2n+1}\le\sqrt{n^2}=n$ for all sufficiently big n

$\displaystyle 0\le\underbrace{\sum_{n=1}^{\infty}\frac{1}{n}}_{\text{harmonic}}\le\sum_{n=1}^{\infty}\frac{1}{\sqrt{2n+1}}\rightarrow$ diverges]

If $x=-1$ then $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{2n+1}}$ is convergent by AST

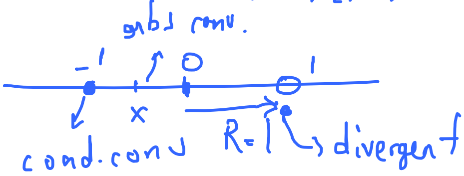
condition. conv.

[• it's an alter. series

• $\displaystyle\lim_{n\to\infty}\frac{1}{\sqrt{2n+1}}=0$

• $\displaystyle\frac{1}{\sqrt{2n+3}}\le\frac{1}{\sqrt{2n+1}}$ ✓]

Interval of convergence is $[-1,1)$



1b. $\displaystyle\sum_{n=2}^{\infty} \frac{(2x-1)^n}{n\ln n} = \sum_{n=2}^{\infty} \frac{2^n\left(x-\frac{1}{2}\right)^n}{n\ln(n)} \rightsquigarrow c = \frac{1}{2}$

$\displaystyle\lim_{n\to\infty} \frac{\ln n}{\ln(n+1)} \overset{L'H}{=} 1$

$\sum a_n (x-c)^n$

(Absolute) Ratio test

$$\lim_{n\to\infty} \left| \frac{2^{n+1}\left(x-\frac{1}{2}\right)^{n+1}}{(n+1)\ln(n+1)} \cdot \frac{n\ln(n)}{2^n\left(x-\frac{1}{2}\right)^n} \right| = \lim_{n\to\infty} \left| \frac{2\ln(n)}{\ln(n+1)} \cdot \frac{n}{n+1} \cdot \left(x-\frac{1}{2}\right) \right| = 2\left|x-\frac{1}{2}\right| < 1$$

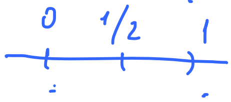
$$\Downarrow$$

$$\left|x-\frac{1}{2}\right| < \frac{1}{2}$$

radius.

Endpoints

If $x=0$ then $\displaystyle\sum_{n=2}^{\infty} \frac{2^n\left(-\frac{1}{2}\right)^n}{n\ln(n)} = \sum_{n=2}^{\infty} \frac{(-1)^n}{n\ln(n)}$ is cond. convergent.



Now, $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\ln(n)}$ diverges.

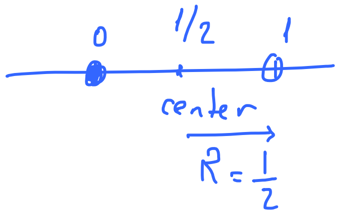
$\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{n\ln(n)}$ is convergent by AST.

If $x=1$ then $\displaystyle\sum_{n=2}^{\infty} \frac{2^n \cdot \frac{1}{2^n}}{n\ln(n)} = \sum_{n=2}^{\infty} \frac{1}{n\ln n} \to$ divergent.

Integral test

$\displaystyle\int_2^{\infty} \frac{dx}{x\ln x}$ $u = \ln x$

Interval of convergence $[0, 1)$



$1c$ $\displaystyle\sum_{n=1}^{\infty}\left(\frac{n}{n+1}\right)^{n^2}x^n \rightsquigarrow \underline{c=0}$

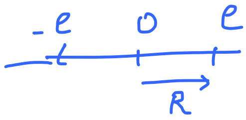
Ratio test – HW

$$\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n = \frac{1}{\lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n} = \frac{1}{e}$$

Soln: Root test

$$\lim_{n\to\infty}\left|\left(\frac{n}{n+1}\right)^n x\right| = \frac{1}{e}|x| < 1 \Rightarrow |x| < e$$

$R = e$



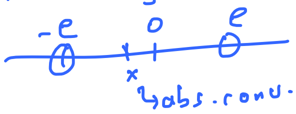
Endpoints:

If $x=e$ then $\displaystyle\sum_{n=1}^{\infty}\left(\frac{n}{n+1}\right)^{n^2}e^n = \underline{\text{divergent}}$ by LCT.
by general term test.

If $x=-e$ then $\displaystyle\sum_{n=1}^{\infty}\left(\frac{n}{n+1}\right)^{n^2}(-1)^n e^n = \text{diverge.}$

Thus, the endpoints $x=e$ and $x=-e$ donot belong to the interval of convergence.

Interval of convergence $(-e, e)$



$$\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^{n^2}e^n = ? = e^{1/2} \neq 0 \longrightarrow \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^{n^2}(-1)^n e^n = \underline{\underline{DNE}}$$

Log. limits

$\rightarrow y = \left(\frac{x}{x+1}\right)^{x^2}e^x$

$\lim y = e^{1/2}$

$\ln y = x^2 \ln\left(\frac{x}{x+1}\right) + x$

$$\lim_{x\to\infty}\left[x^2\ln\left(\frac{x}{x+1}\right)+x\right] = \lim_{x\to\infty}\left[\frac{\ln\left(\frac{x}{x+1}\right)+\frac{1}{x}}{1/x^2}\right] \overset{LH}{=} \lim_{x\to\infty}\frac{-\frac{1}{x^3+x}}{-2/x^3} = \frac{1}{2} = \lim_{x\to\infty}\ln y.$$

Q2: Find a power series representation of the following functions centered at the given points.

2a $f(x) = \dfrac{2}{4-x}$, $c = -2$

$\left[\dfrac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \ |x| < 1 \right]$

$\dfrac{2}{4-x}$ not defined when $x = 4$

$-2 \quad 4$ radius is at most 6.

6

Soln:

So, we expand $\dfrac{2}{4-x}$ in powers of $(x+2)$.

$$\frac{2}{4-x} = \frac{2}{6-(x+2)} = \frac{2}{6\left[1-\left(\frac{x+2}{6}\right)\right]} = \frac{2}{6} \frac{1}{1-\frac{x+2}{6}} = \frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{x+2}{6}\right)^n \text{ whenever } \left|\frac{x+2}{6}\right| < 1$$

$$= \sum_{n=0}^{\infty} \frac{(x+2)^n}{3 \cdot 6^n} \text{ whenever } |x+2| < 6$$

$c = -2$

$-6 < x+2 < 6$

$-8 < x < 4$

$-8 \quad -2 \quad 4$

I.C $= (-8, 4)$

Endpoints:

If $x = 4$, $\dfrac{1}{3}\sum_{n=0}^{\infty} \dfrac{6^n}{6^n} = \dfrac{1}{3}\sum_{n=0}^{\infty} 1 =$ diverges

If $x = -8$, $\dfrac{1}{3}\sum_{n=0}^{\infty} \dfrac{(-6)^n}{6^n} = \dfrac{1}{3}\sum_{n=0}^{\infty} (-1)^n$ diverges since $\lim_{n\to\infty} (-1)^n =$ DNE

So, the interv. of convergence is $(-8, 4)$, $c = -2$, $R = 6$

Extra:

Center	Power series expansion of $\dfrac{2}{4-x}$	Radius.
$c = 0$	$\dfrac{1}{2} + \dfrac{x}{8} + \dfrac{x^2}{32} + \dfrac{x^3}{128} + \cdots$	R is at most 4
$c = -2$	$\dfrac{1}{3} + \boxed{\dfrac{x+2}{18}} + \boxed{\dfrac{(x+2)^2}{108}} + \dfrac{(x+2)^3}{648} + \cdots$	$R = 6$
$c = 3$	$2 + 2(x-3) + 2(x-3)^2 + 2(x-3)^3 + \cdots$	R is at most 1.

$x = 4$

$0 \quad 4$

$-2 \quad 4$

$3 \quad 4$

$\bullet \ \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad |x|<1$

2b: $f(x) = \ln(2-x)$, $c=0$ $\quad 2-x>0$

$$\frac{d}{dx}\ln(2-x) = \frac{-1}{2-x} = -\frac{1}{2}\,\frac{1}{1-\frac{x}{2}} = -\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{x}{2}\right)^n \text{ whenever } \left|\frac{x}{2}\right|<1$$

Definite Integral Approach

$$\int_0^x \frac{-dt}{2-t} = \int_0^x -\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{t}{2}\right)^n dt = -\frac{1}{2}\sum_{n=0}^{\infty}\int_0^x \frac{t^n}{2^n}dt = -\frac{1}{2}\sum_{n=0}^{\infty}\frac{t^{n+1}}{2^n(n+1)}\Bigg|_{t=0}^{t=x} \qquad |x|<2$$

So,
$$\int_0^x \frac{-dt}{2-t} = -\frac{1}{2}\sum_{n=0}^{\infty}\frac{x^{n+1}}{2^n(n+1)}$$

Also,
$$\int_0^x \frac{-dt}{2-t} = \int_2^{2-x}\frac{du}{u} = \ln|u|\Big|_{u=2}^{2-x} = \ln(2-x) - \ln(2) = -\frac{1}{2}\sum_{n=0}^{\infty}\frac{x^{n+1}}{2^n(n+1)}$$

$u = 2-t$, $du = -dt$

So,
$$\ln(2-x) = \ln 2 - \frac{1}{2}\sum_{n=0}^{\infty}\frac{x^{n+1}}{2^n(n+1)} = \ln 2 - \frac{1}{2}\left[x + \frac{x^2}{4} + \frac{x^3}{12} + \cdots\right] \qquad |x|<2.$$

Endpoint: If $x=2$ then $-\frac{1}{2}\sum_{n=0}^{\infty}\frac{2}{n+1}$ diverges

If $x=-2$ then $-\frac{1}{2}\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{n+1}$ converges by AST.

So, interval of convergence is $[-2,2)$
$R=2$
$c=0$

Indefinite Integral Approach:

$$\frac{d}{dx}\ln(2-x) = -\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{x}{2}\right)^n, \quad |x|<2. \text{ Thus, } \int -\frac{1}{2}\sum_{n=0}^{\infty}\frac{x^n}{2^n}dx = -\frac{1}{2}\sum_{n=0}^{\infty}\frac{x^{n+1}}{2^n(n+1)} + C = \ln(2-x)$$

Put $x=0$ then $C = \ln 2$.

So,
$$\ln(2-x) = \ln 2 - \frac{1}{2}\sum_{n=0}^{\infty}\frac{x^{n+1}}{2^n(n+1)}, \quad |x|<2.$$

Q3: Find the sum of the following series.

3a: $\displaystyle\sum_{n=2}^{\infty} \frac{n(n-1)}{2^n}$ not a power series

Soln: $\displaystyle\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}$, $|x| < 1$

$\displaystyle\sum_{n=1}^{\infty} n x^{n-1} = 1 + 2x + 3x^2 + \cdots = \frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{1}{(1-x)^2}$, $|x| < 1$

$\displaystyle\sum_{n=2}^{\infty} n(n-1) x^{n-2} = 2 + 6x + 12x^2 + \cdots = \frac{d^2}{dx^2}\left(\frac{1}{1-x}\right) = \frac{2}{(1-x)^3}$, $|x| < 1$.

$$\sum_{n=2}^{\infty} \frac{n(n-1)}{2^n} = \frac{1}{4} \sum_{n=2}^{\infty} \frac{n(n-1)}{2^{n-2}} = \frac{1}{4}\left[\frac{2}{(1-x)^3}\right]\Bigg|_{x=\frac{1}{2}} = \frac{1}{4}\,\frac{2}{\left(1-\frac{1}{2}\right)^3} = 4$$

3b: $\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1}}{4^{2n-1}(2n+1)!}$

Soln: In the next chapter, $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$, $x \in \mathbb{R}$.

So,

$$\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1}}{4^{2n-1}\cdot(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1}}{4^{2n+1}\cdot 4^{-2}\cdot(2n+1)!} = 16\sin\frac{\pi}{4} = 16\,\frac{\sqrt{2}}{2} = 8\sqrt{2}.$$

Q4: Use the power series of $\ln\frac{1+x}{1-x}$ to find the value of $\sum_{k=0}^{\infty}\frac{2}{(2k+1)3^{2k+1}}$.

Soln: $\ln\left(\frac{1+x}{1-x}\right)=\ln(1+x)-\ln(1-x)$

Step#1: Power series of $\ln(1+x)$.

$\frac{1}{1-x}=\sum_{n=0}^{\infty}x^n,\ |x|<1$

$\frac{1}{1+x}=\sum_{n=0}^{\infty}(-1)^n x^n\quad |-x|=|x|<1 \Rightarrow \ln(1+x)=\int\frac{dx}{1+x}=\int\sum_{n=0}^{\infty}(-1)^n x^n dx=\sum_{n=0}^{\infty}(-1)^n\int x^n dx=\sum_{n=0}^{\infty}(-1)^n\frac{x^{n+1}}{n+1}+C$

$C=0$, $\ln 1 = 0$

$=\sum_{n=0}^{\infty}(-1)^n\frac{x^{n+1}}{n+1}=\sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^n}{n}$

Step#2: For $\ln(1-x)$, put $-x$ in

$\ln(1-x)=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}(-1)^n x^n}{n}=\sum_{n=1}^{\infty}(-1)^{2n+1}\frac{x^n}{n}=-\sum_{n=1}^{\infty}\frac{x^n}{n}$ $\quad |x|<1$

Step#3:

$\ln\left(\frac{1+x}{1-x}\right)=\sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^n}{n}+\sum_{n=1}^{\infty}\frac{x^n}{n}=\sum_{n=1}^{\infty}\left[(-1)^{n+1}+1\right]\frac{x^n}{n}=2\sum_{k=0}^{\infty}\frac{x^{2k+1}}{2k+1}$

If n is even, $(-1)^{n+1}+1=0$

odd, $(-1)^{n+1}+1=2$

In order to get $\sum_{k=0}^{\infty}\frac{2}{(2k+1)3^{2k+1}}$, x should be $\frac{1}{3}$

So, $\sum_{k=0}^{\infty}\frac{2}{(2k+1)3^{2k+1}}=\ln\left(\frac{1+1/3}{1-1/3}\right)=\ln\left(\frac{4/3}{2/3}\right)=\ln 2$

Q5: Find the power series for $\frac{\ln(1-x)}{1-x}$ at $c=0$.

Soln:

Fact#1: If $f(x)=\sum_{n=0}^{\infty} a_n x^n$ for $|x|<1$ then $\frac{f(x)}{1-x}=\sum_{n=0}^{\infty}\left[a_0+a_1+\cdots+a_n\right]x^n$, $|x|<1$.

Verification:

$$\frac{1}{1-x}\left[a_0+a_1x+a_2x^2+\cdots\right]=(1+x+x^2+x^3+\cdots)(a_0+a_1x+a_2x^2+\cdots)$$

$$=a_0+(a_0+a_1)x+(a_0+a_1+a_2)x^2+\cdots \quad \boxtimes$$

From the prev. question, $\ln(1-x)=-\sum_{n=1}^{\infty}\frac{x^n}{n}=-x-\frac{x^2}{2}-\frac{x^3}{3}-\cdots$

From Fact#1, $\frac{\ln(1-x)}{1-x}=-\sum_{n=1}^{\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right)x^n$. $|x|<1$. $\boxtimes$