Math 118. Chapter 9.4: Absolute and Conditional Convergence

Note Title

Q1: Determine whether the series converges absolutely, converge conditionally or diverge

1a: $\sum_{n=1}^{\infty} (-1)^n \sin\left(\frac{1}{2n}\right)$

Soln: Consider $\sum_{n=1}^{\infty} \sin\left(\frac{1}{2n}\right)$.

Compare $\sum_{n=1}^{\infty} \sin\left(\frac{1}{2n}\right)$ with $\sum_{n=1}^{\infty} \frac{1}{2n}$ $\nearrow$ $\frac{1}{2}\times$ harmonic series diverges.

$$\lim_{n\to\infty} \frac{\sin\left(\frac{1}{2n}\right)}{\frac{1}{2n}} = \lim_{w\to 0^+} \frac{\sin w}{w} = 1 \begin{matrix} <\infty \\ >0 \end{matrix}$$

$w = \frac{1}{2n}$

$n\to\infty \Rightarrow w \to 0^+$

So, $\sum_{n=1}^{\infty} \sin\left(\frac{1}{2n}\right)$ diverges.

So, $\sum_{n=1}^{\infty} (-1)^n \sin\left(\frac{1}{2n}\right)$ is <u>NOT</u> absolutely convergent.

Check if $\sum_{n=1}^{\infty} (-1)^n \sin\left(\frac{1}{2n}\right)$ converges?. Use the AST

- it's an alternating series
- $\frac{1}{2n+2} \le \frac{1}{2n}$ for $n \geq 1$.

$y = \sin x$ is increasing

$\sin\left(\frac{1}{2n+2}\right) \le \sin\left(\frac{1}{2n}\right)$

- $\lim_{n\to\infty} \sin\left(\frac{1}{2n}\right) = \sin(0) = 0$

So, $\sum_{n=1}^{\infty} (-1)^n \sin\left(\frac{1}{2n}\right)$ converges

So, it's conditionally convergent.

1b. $\sum_{n=1}^{\infty} \frac{\sin(n)\, 118^n}{n!}$ — not a positive series
— not an alternating series

$$0 \le \left| \frac{\sin(n)\, 118^n}{n!} \right| \le \frac{118^n}{n!}$$

$$\sum_{n=1}^{\infty} 0 = 0 \le \underbrace{\sum_{n=1}^{\infty} \left| \frac{\sin(n) \cdot 118^n}{n!} \right|}_{\text{converges}} \le \sum_{n=1}^{\infty} \frac{118^n}{n!}$$

converges by the Ratio test

$$\left[\lim_{n\to\infty} \frac{118^{n+1}}{(n+1)!} \cdot \frac{n!}{118^n} = \lim_{n\to\infty} \frac{118}{n+1} = 0 < 1 \right]$$

So, $\sum_{n=1}^{\infty} \frac{\sin(n) \cdot 118^n}{n!}$ is absolutely convergent, and hence, convergent but not conditionally convergent.

1c: $\sum_{n=1}^{\infty} \frac{n!}{(-100)^n} = \sum_{n=1}^{\infty} (-1)^n \frac{n!}{100^n}$

• The series $\sum_{n=1}^{\infty} \frac{n!}{100^n}$ diverges $\left[\lim_{n\to\infty} \frac{n!}{100^n} = \infty \text{ , OR, } \lim_{n\to\infty} \frac{(n+1)!}{100^{n+1}} \cdot \frac{100^n}{n!} = \lim_{n\to\infty} \frac{n+1}{100} = \infty > 1 \text{. Ratio} \right]$

So, $\sum_{n=1}^{\infty} (-1)^n \frac{n!}{100^n}$ is NOT abs. convergent

$\lim_{n\to\infty} \frac{n!}{100^n} = \infty$, so, we CANNOT use AST.

$\lim_{n\to\infty} \frac{(-1)^n n!}{100^n} = \text{DNE}$, so, $\sum_{n=1}^{\infty} \frac{(-1)^n n!}{100^n}$ diverges. So, it's not conditionally convergent.

1d: $\sum_{i=1}^{\infty} \frac{(-1)^{i+1}}{i\pi^i}$

$\pi = 3.1415926\ldots$

Soln: Consider $\sum_{i=1}^{\infty} \frac{1}{i\pi^i}$. We have $\pi^i \leq i\pi^i$ whenever $i \geq 1$.

So, $0 \leq \frac{1}{i\pi^i} \leq \frac{1}{\pi^i}$

So, $0 \leq \underbrace{\sum_{i=1}^{\infty} \frac{1}{i\pi^i}}_{\text{convergent.}} \leq \sum_{i=1}^{\infty} \frac{1}{\pi^i} \rightarrow$ geometric series $r = \frac{1}{\pi} < 1$ converges

Thus, $\sum_{i=1}^{\infty} \frac{(-1)^{i+1}}{i\pi^i}$ is absolutely convergent.

is NOT conditionally convergent

is convergent.

is NOT divergent.

Ie: $\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{(n+1)\ln(n+1)} = \sum_{n=1}^{\infty} \frac{(-1)^n}{(n+1)\ln(n+1)}$

Consider $\sum_{n=1}^{\infty} \frac{1}{(n+1)\ln(n+1)} = \sum_{n=2}^{\infty} \frac{1}{n\ln(n)}$. Use the integral test. $f(x) = \frac{1}{x\ln(x)}$ for $x \geq 2$.

positive, $x \geq 2$

The function $y = f(x)$ is continuous and decreasing on $[2,\infty)$ since $f'(x) = -\frac{\ln x + 1}{x^2(\ln x)^2} < 0$

$$\int_2^{\infty} \frac{dx}{x\ln x} = \lim_{M\to\infty} \underbrace{\int_2^{M} \frac{dx}{x\ln x}}_{\begin{array}{c} w = \ln x \\ dw = \frac{dx}{x} \\ \frac{x=2}{x=M} \end{array}} = \lim_{M\to\infty} \int_{\ln 2}^{\ln M} \frac{dw}{w} = \lim_{M\to\infty} \ln|w| \Big|_{w=\ln 2}^{\ln M} = \lim_{M\to\infty} \Big[\ln|\ln M| - \ln|\ln 2|\Big]$$

$= \infty$.

So, by the integral test, $\sum_{n=2}^{\infty} \frac{1}{n\ln(n)}$ diverges.

So, $\sum_{n=1}^{\infty} \frac{(-1)^n}{(n+1)\ln(n+1)}$ is not absolutely convergent.

Use AST

- is an alternating series ✓
- $\frac{1}{(n+2)\ln(n+2)} \leq \frac{1}{(n+1)\ln(n+1)}$
- $\lim_{n\to\infty} \frac{1}{(n+1)\ln(n+1)} = 0$

Obv.

By AST, $\sum_{n=1}^{\infty} \frac{(-1)^n}{(n+1)\ln(n+1)}$ converges.

Thus, it's conditionally convergent.

Q2: Show that the series $\sum_{n=1}^{\infty} a_n$ converges absolutely where $a_n = \frac{10}{n^2}$ for even n and $a_n = -\frac{1}{10n^3}$ for odd n.

$$a_n = \begin{cases} \frac{10}{n^2} & \text{if } n \text{ is even} \\ -\frac{1}{10n^3} & \text{if } n \text{ is odd.} \end{cases}$$

$\sum_{n=1}^{\infty} a_n$ $\quad n=1 \cdot n=2$

m terms

$$\sum_{n=1}^{\infty} |a_n| = |a_1| + |a_2| + |a_3| + \cdots = \frac{1}{10} + \frac{10}{2^2} + \frac{1}{10.3^3} + \frac{10}{4^2} + \frac{1}{10.5^3} + \cdots$$

- m^{th} partial sum

Let $S_m = \sum_{n=1}^{m} |a_n|$

$$\sum_{n=1}^{\infty} \frac{10}{n^2} = \frac{10}{1} + \frac{10}{2^2} + \frac{10}{3^2} + \cdots$$

- convergent p-series $p=2$

Let $t_m = \sum_{n=1}^{m} \frac{10}{n^2}$

$$\sum_{n=1}^{\infty} \frac{1}{10n^3} = \frac{1}{10} + \frac{1}{10.2^3} + \frac{1}{10.3^3} + \cdots$$

$\to$ convergent $p=3$

Let $w_m = \sum_{n=1}^{m} \frac{1}{10n^3}$

Now, $0 \le S_m \le t_m + w_m$ ✓ for all m,

$$\lim_{m\to 0} 0 = 0 \le \underbrace{\lim_{m\to\infty} S_m}_{\text{exists [also finite]}} \le \lim_{m\to\infty}(t_m + w_m) = \lim_{m\to\infty} t_m + \lim_{m\to\infty} w_m$$

Thus, $\sum_{n=1}^{\infty} |a_n| = \lim_{m\to\infty} S_m$ exists.

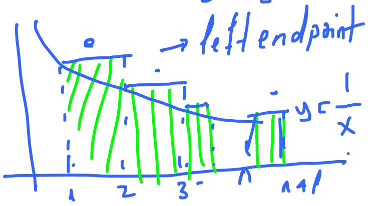
Thus, $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

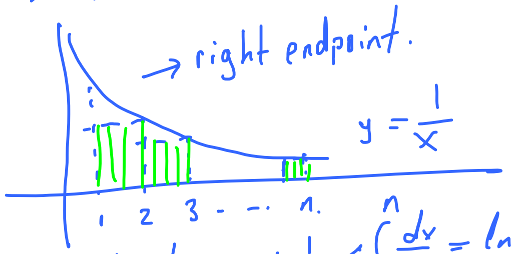
Q3: Show that $\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{n}}$ is conditionally convergent

Consider $\sum_{n=1}^{\infty} \frac{1}{1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{n}}$.

Claim #1: $\ln(n+1) \le 1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{n} \le 1+\ln(n)$ for all $n \ge 1$.

Verification:





$$\int_1^{n+1} \frac{dx}{x} \le 1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{n}$$

$$\ln(n+1) \le 1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{n}$$

$$\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{n} \le \int_1^n \frac{dx}{x} = \ln n$$

$$1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{n} \le 1+\ln n.$$

Step #2 LCT. Compare $\sum_{n=2}^{\infty} \frac{1}{1+\frac{1}{2}+\dots+\frac{1}{n}}$ with $\sum_{n=2}^{\infty} \frac{1}{\ln n}$ → diverges since $\ln n \le n$ for $n \ge 2$

$$0 \le \underbrace{\sum_{n=2}^{\infty} \frac{1}{n}}_{\text{harmonic}} \le \sum_{n=2}^{\infty} \frac{1}{\ln n} \to \text{is divergent.}$$

$$\lim_{n\to\infty} \frac{\frac{1}{\ln n}}{\frac{1}{1+\frac{1}{2}+\dots+\frac{1}{n}}} = \lim_{n\to\infty} \frac{1+\frac{1}{2}+\dots+\frac{1}{n}}{\ln n} = 1 \begin{array}{l} < \infty \\ > 0 \end{array}$$

From Claim #1,
$$\underbrace{\lim_{n\to\infty} \frac{\ln(n+1)}{\ln n}}_{} \le \underbrace{\lim_{n\to\infty} \frac{1+\frac{1}{2}+\dots+\frac{1}{n}}{\ln n}}_{= 1} \le \underbrace{\lim_{n\to\infty} \frac{1+\ln(n)}{\ln(n)}}_{}$$

$$\lim_{x\to\infty} \frac{\ln(x+1)}{\ln x} = \lim_{x\to\infty} \frac{1/(x+1)}{1/x} = 1. \qquad \lim_{x\to\infty} \frac{1+\ln x}{\ln x} = \lim_{x\to\infty} \frac{1/x}{1/x} = 1.$$

So, $\sum_{n=2}^{\infty} \frac{1}{1+\frac{1}{2}+\dots+\frac{1}{n}}$ is divergent by LCT. Thus, $\sum_{n=2}^{\infty} \frac{(-1)^n}{1+\frac{1}{2}+\dots+\frac{1}{n}}$ is not abs. convergent.

Step #3. Use AST.

- alternating series.
- $\frac{1}{1+\frac{1}{2}+\dots+\frac{1}{n}+\frac{1}{n+1}} \le \frac{1}{1+\frac{1}{2}+\dots+\frac{1}{n}}$

Claim #1.

$$0 \le \lim_{n\to\infty} \frac{1}{1+\frac{1}{2}+\dots+\frac{1}{n}} \le \lim_{n\to\infty} \frac{1}{\ln(n+1)} = 0$$

By AST, $\sum_{n=2}^{\infty} \frac{(-1)^n}{1+\frac{1}{2}+\dots+\frac{1}{n}}$ converges. So, it's conditionally convergent.

Q4: How many terms of the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+n}$ are needed to compute the sum of the series with error less than 0.001?

Soln: $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+n}$ satisfies the conditions given in AST. Indeed,

- it's alternating
- $\frac{1}{(n+1)^2+n+1} \le \frac{1}{n^2+n}$
- $\lim_{n\to\infty} \frac{1}{n^2+n} = 0.$

Thus, this series is convergent. Now,

$$|s - s_n| \le |a_{n+1}|$$

So, if $\frac{1}{(n+1)^2+(n+1)} < \frac{1}{1000}$ $\left[\text{i.e, } (n+1)^2+n+1 > 1000\right]$ then the error is less than 0.001.

$\boxed{n = 31}$

So, we need 31 terms of the series to compute the sum within 0.001 of its actual value.

$\boxed{/\!/}$

0.001

By a calculator, excel

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+n} \approx \boxed{-0.3862943...}$$

31^{st} partial sum $\sum_{n=1}^{31} \frac{(-1)^n}{n^2+n} \approx \boxed{-0.3867824...}$

30^{th} partial sum $\sum_{n=1}^{30} \frac{(-1)^n}{n^2+n} \approx -0.38577...$