Q1: Show that the following series diverge.

a. $$\sum_{n=1}^{\infty} \frac{(-1)^n (n^2+n)}{4n^2+5n-1}$$

Soln: $\lim_{n\to\infty} \frac{n^2+n}{4n^2+5n-1} = \frac{1}{4} \neq 0$. So, we can't use AST.

We can't say that the series diverges.

Instead,

$\lim_{n\to\infty} (-1)^n \frac{n^2+n}{4n^2+5n-1} = DNE \neq 0$

The given series diverges. [General term test]

AST: $\sum_{n=1}^{\infty} (-1)^{n-1} a_n = a_1 - a_2 + a_3 - \cdots$

(i) $a_n \geq 0$

(ii) $a_n \geq a_{n-1}$ for all large n.

(iii) $a_n \to 0$ as $n \to \infty$

Then $\sum_{n=1}^{\infty} (-1)^{n-1} a_n$ is convergent.

Fact #1: If $\sum_{n=1}^{\infty} (-1)^{n-1} a_n$ diverges then at least one of i, ii, iii fails.

Fact #2: You may have a convergent alternating series for which AST fails.

Fact #3: If ii or iii fails, you can't conclude that the alternating series diverges.

1b $\sum_{n=1}^{\infty} \frac{n!}{100^n} \to a_n = \frac{n!}{100^n}$

Tech #2: $\lim_{n\to\infty} \frac{n!}{100^n} = \infty$ General term test.

Ratio Test

$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)!}{100^{n+1}} \cdot \frac{100^n}{n!} = \lim_{n\to\infty} \frac{n+1}{100} = \infty > 1$. So, the given series diverges.

radian

$\overline{1, 2, 3}^{\,n}$ $\overline{4, 5, 6, \ldots}^{\,n}$

sin values are negative.

1c. $\sum_{n=1}^{\infty} \sin(n)$ → is not alternating
→ is not a positive series.

Soln: $\lim_{n\to\infty} \sin(n) = DNE$. So, the given series diverges.
by the gen. term test.

(2) Use a comparison test to test the following series for convergence.

2a $\sum_{n=1}^{\infty} \frac{1}{2n^3 - n}$

Comparison Test:

Observe $n^2 \le 2n^3 - n$ for big n.

$2n^3 - n^2 - n \geq 0$ for big n

So, $0 \le \frac{1}{2n^3 - n} \le \frac{1}{n^2}$ for sufficiently big n.

$$0 \le \underbrace{\sum_{n=1}^{\infty} \frac{1}{2n^3 - n}}_{\text{this series also converges.}} \le \underbrace{\sum_{n=1}^{\infty} \frac{1}{n^2}}_{\substack{\text{p-series} \\ p = 2 > 1. \\ \text{So, it converges.}}}$$

Limit Comparison:

Compare $\sum_{n=1}^{\infty} \frac{1}{2n^3 - n}$ ≥ 0 with $\sum_{n=1}^{\infty} \frac{1}{n^3}$ ≥ 0 p-series $p = 3$ converges.

$$\lim_{n \to \infty} \frac{\frac{1}{2n^3 - n}}{\frac{1}{n^3}} = \lim_{n \to \infty} \frac{n^3}{2n^3 - n} = \frac{1}{2} > 0 \quad < \infty$$

So, $\sum_{n=1}^{\infty} \frac{1}{2n^3 - n}$ converges.

2b: $\sum_{n=1}^{\infty} \arcsin 2^{-n}$

MVT $\quad \sin x \le x$

$x \le \arcsin x \le 2x$

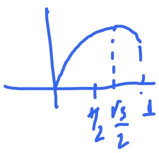
Comparison:

Claim: $\arcsin x \le 2x$ if $x>0$, x is close to 0.

Verification: $f(x) = 2x - \arcsin x$, $f(0) = 0 - \arcsin 0 = 0$

$f'(x) = 2 - \frac{1}{\sqrt{1-x^2}}$, $f'(x) = 0 = 2\sqrt{1-x^2} - 1$

$x = \frac{\sqrt{3}}{2}$.



$2x \ge \arcsin x$

So, $0 \le \arcsin 2^{-n} \le 2 \cdot 2^{-n}$ if $n \ge 1$

$$0 \le \sum_{n=1}^{\infty} \arcsin 2^{-n} \le \underbrace{2\sum_{n=1}^{\infty} 2^{-n}}_{\text{is convergent}} = 2\left(\frac{1}{2}\left(1+\frac{1}{2}+\frac{1}{4}+\cdots\right)\right) = 2 \cdot \frac{1}{2} \cdot \frac{1}{1-1/2} = 2.$$

By comparison,
this series is also convergent.

$\lim_{x \to 0^+} \frac{\sin x}{x} = 1 \quad \sin x \approx x$

$x \approx \arcsin x$

Limit Comparison

Compare $\sum_{n=1}^{\infty} \arcsin 2^{-n}$ with $\sum_{n=1}^{\infty} \frac{1}{2^n}$?

Geo. series convergent

$$\lim_{n\to\infty} \frac{\frac{1}{2^n}}{\arcsin \frac{1}{2^n}}$$

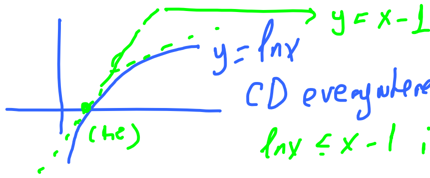
$$\lim_{x\to\infty} \frac{\frac{1}{2^x}}{\arcsin \frac{1}{2^x}} = \lim_{x\to\infty} \frac{-2^{-x}\ln 2}{\frac{-2^{-x}\ln 2}{\sqrt{1-2^{-2x}}}}$$

$$= \lim_{x\to\infty} \sqrt{1-2^{-2x}} = 1 \quad {<\infty \atop >0}$$

So, $\sum_{n=1}^{\infty} \arcsin 2^{-n}$ converges.

2c

$$\sum_{k=2}^{\infty} \frac{1}{\ln k}$$



CD everywhere.

$\ln x \le x - 1$ if $x \geq 1$

Comparison:

$\ln k \le k - 1$ if $k \geq 2$

So, $0 \le \frac{1}{k-1} \le \frac{1}{\ln k}$ if $k \geq 2$

$$0 \le \underbrace{\sum_{k=2}^{\infty} \frac{1}{k-1}}_{\text{harmonic series diverges.}} \le \underbrace{\sum_{k=2}^{\infty} \frac{1}{\ln k}}_{\text{This series diverges.}}$$

Limit Comparison

harmonic. diverges

Compare $\sum_{k=2}^{\infty} \frac{1}{\ln k}$ with $\sum_{k=2}^{\infty} \frac{1}{k}$

$$\lim_{k \to \infty} \frac{1/k}{1/\ln k} = \lim_{k \to \infty} \frac{\ln k}{k} = 0$$

So, $\sum_{k=2}^{\infty} \frac{1}{k}$ diverges $\Rightarrow \sum_{k=2}^{\infty} \frac{1}{\ln k}$ diverges.

③ Test the following series for convergence.

3a: $\displaystyle\sum_{n=1}^{\infty} \frac{118n^4 - 117n^2 + 5}{\sqrt[4]{n^{17} + n - 1}}$

$\displaystyle\sum_{n=1}^{\infty} \frac{n^4}{n^{17/4}} = \sum \frac{1}{n^{17/4 - 4}} = \sum \frac{1}{n^{1/4}}$

Soln: Limit Comparison

Compare the given series with $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{1/4}}$ p-series $p = \frac{1}{4} < 1$ diverges.

$$\lim_{n\to\infty} \frac{1/n^{1/4}}{\dfrac{118n^4 - 117n^2 + 5}{\sqrt[4]{n^{17}+n-1}}} = \lim_{n\to\infty} \frac{\sqrt[4]{n^{17}+n-1}}{n^{1/4}(118n^4 - 117n^2 + 5)} = \lim_{n\to\infty} \frac{\cancel{n^{17/4}}\sqrt[4]{1 + \frac{1}{n^{16}} - \frac{1}{n^{17}}}}{\cancel{n^{17/4}}\left(118 - 117n^{2-\frac{17}{4}} + \frac{5}{n^{17/4}}\right)}$$

$$= 1 \quad \begin{matrix} <\infty \\ >0 \end{matrix}$$

So, the given series diverges

3b: $\sum_{n=1}^{\infty} n e^{-n}$

Ratio test

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)e^{-n-1}}{n e^{-n}} = \lim_{n\to\infty} \frac{n+1}{e n} = \frac{1}{e} < 1$$

So, this series is convergent.

3c: $\sum_{n=1}^{\infty} \frac{(2n)!}{3^n (n!)^2}$

Ratio test

$$\lim_{n\to\infty} \frac{(2n+2)!}{3^{n+1} \cdot ((n+1)!)^2} \cdot \frac{3^n (n!)^2}{(2n)!} = \lim_{n\to\infty} \frac{(2n+1)(2n+2)}{3(n+1)^2} = \frac{4}{3} > 1$$

So, the given series diverges.

3d: $\underbrace{\sum_{n=1}^{\infty}\left(1+\frac{2}{n}\right)^{n^2}}_{a_n}$?

Tech#1 $\lim_{n\to\infty}\left(1+\frac{2}{n}\right)^{n^2} \neq 0$ diverges.

Tech #2. Root test

$\underbrace{\lim_{n\to\infty}\left(1+\frac{2}{n}\right)^{n}}_{a_n^{1/n}} = \left(\lim_{n\to\infty}\left(1+\frac{1}{\frac{n}{2}}\right)^{\frac{n}{2}}\right)^{2} = e^2 > 1$

e

So, the given series diverges.

Fact: ★★

$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n}$

$\boxed{\text{Log. Limits.}}$

$(e^x)' = e$

$1 = \lim_{h\to 0}\frac{e^h-1}{h}$

$e^h - 1 \approx h$

$e \approx (h+1)^{1/h}$

$e \approx \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n}$

3e: $\sum_{n=1}^{\infty}\frac{n^2+1}{n^2-3}$

$\lim_{n\to\infty}\frac{n^2+1}{n^2-3} = 1 \neq 0$. By the general term test, the series diverges.