Q1: Compute the value of the series $\sum_{n=2}^{\infty} \frac{1}{n^2+2n}$

Soln: $\frac{1}{n^2+2n} = \frac{1}{n(n+2)} = \frac{A}{n} + \frac{B}{n+2} \Rightarrow 1 = A(n+2) + Bn$

$A+B=0$, $2A=1$ $\Rightarrow$ $B=-\frac{1}{2}$, $A=\frac{1}{2}$

So, $\frac{1}{n^2+2n} = \frac{1}{2}\frac{1}{n} - \frac{1}{2}\frac{1}{n+2}$

$n=2$ $\quad a_2 = \frac{1}{2}\frac{1}{2} - \frac{1}{2}\cancel{\frac{1}{4}}$

$n=3$ $\quad a_3 = \frac{1}{2}\frac{1}{3} - \frac{1}{2}\cancel{\frac{1}{5}}$

$n=4$ $\quad a_4 = \frac{1}{2}\cancel{\frac{1}{4}} - \frac{1}{2}\frac{1}{6}$

$n=5$ $\quad a_5 = \frac{1}{2}\cancel{\frac{1}{5}} - \frac{1}{2}\frac{1}{7}$

$\vdots$

$m-1$ $\quad a_{m-1} = \frac{1}{2}\cancel{\frac{1}{m-1}} - \frac{1}{2}\frac{1}{m+1}$

m $\quad a_m = \frac{1}{2}\cancel{\frac{1}{m}} - \frac{1}{2}\frac{1}{m+2}$

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$$S_m = \frac{1}{2}\frac{1}{2} + \frac{1}{2}\frac{1}{3} - \frac{1}{2}\frac{1}{m+1} - \frac{1}{2}\frac{1}{m+2} = \frac{5m^2+3m-8}{12(m+1)(m+2)}$$

$$\sum_{n=2}^{\infty} \frac{1}{n(n+2)} = \lim_{m\to\infty} S_m = \lim_{m\to\infty} \frac{5m^2+3m-8}{12(m+1)(m+2)} = \frac{5}{12}$$

$\downarrow$

This series is convergent.

Remarks

(i) This series is an example of convergent telescoping series.

(ii) Ratio/Root test are inconclusive. HW.

Q2: If exists find the sum of the following series. If not, explain why the series diverges.

2a: $$\sum_{n=2}^{\infty} \frac{2^{2n}+3^n-5}{7^{n+2}} = \frac{1}{49} \sum_{n=2}^{\infty} \frac{2^{2n}+3^n-5}{7^n} = \frac{1}{49}\left[\sum_{n=2}^{\infty} \frac{4^n}{7^n} + \sum_{n=2}^{\infty} \frac{3^n}{7^n} - 5\sum_{n=2}^{\infty} \frac{1}{7^n}\right]$$

[it's indeed the case]

can do this as long as the RHS is finite.

$$\sum_{n=1}^{\infty} r^{n-1} = 1+r+r^2+\dots = \frac{1}{1-r}$$

$$\sum_{n=2}^{\infty} \frac{4^n}{7^n} = \underbrace{1+\frac{4}{7}+\frac{4^2}{7^2}+\frac{4^3}{7^3}+\frac{4^4}{7^4}+\dots}_{\text{geo. series } r=\frac{4}{7}} -1-\frac{4}{7} = \frac{1}{1-\frac{4}{7}}-1-\frac{4}{7} = \frac{7}{3}-1-\frac{4}{7} = \frac{16}{21}$$

$$\sum_{n=2}^{\infty} \frac{3^n}{7^n} = \frac{3^2}{7^2}+\frac{3^3}{7^3}+\dots = \frac{3^2}{7^2}\underbrace{\left(1+\frac{3}{7}+\frac{3^2}{7^2}+\dots\right)}_{\text{geo. ser. } r=\frac{3}{7}} = \frac{3^2}{7^2}\,\frac{1}{1-\frac{3}{7}} = \frac{9}{49}\,\frac{7}{4} = \frac{9}{28}$$

$$\sum_{n=2}^{\infty} \frac{1}{7^n} = \frac{1}{7^2}+\frac{1}{7^3}+\dots = \frac{1}{7^2}\underbrace{\left(1+\frac{1}{7}+\frac{1}{7^2}+\dots\right)}_{\text{geo. ser, } r=\frac{1}{7}} = \frac{1}{49}\cdot\frac{1}{1-\frac{1}{7}} = \frac{1}{42}$$

$$\text{Answer} = \frac{1}{49}\left[\frac{16}{21}+\frac{9}{28}-\frac{5}{42}\right] = \frac{27}{1372} \quad //$$

2b. $\sum_{n=0}^{\infty} \frac{\pi^n - e^{2n+1}}{5^n}$ diverges. $e^2 \approx 7.38 > 5$

negative.

$$\lim_{n\to\infty} \frac{\pi^n - e^{2n+1}}{5^n} = \lim_{n\to\infty} \frac{e^{2n}\left[\left(\frac{\pi}{e^2}\right)^n - e\right]}{5^n} = -\infty \neq 0$$

So, the given series is divergent since the general term of the series tends to a nonzero #.

2c $\sum_{n=1}^{\infty} \left(1+\frac{2}{n}\right)^{n^2}$ diverges.

$$y = \left(1+\frac{2}{x}\right)^{x^2}$$

$$\ln y = x^2 \ln\left(1+\frac{2}{x}\right)$$

Replace n by x. Find $\lim_{x\to\infty} \left(1+\frac{2}{x}\right)^{x^2} = ?$

$$\lim_{x\to\infty} x^2 \ln\left(1+\frac{2}{x}\right) = \lim_{x\to\infty} \frac{\ln\left(1+\frac{2}{x}\right)}{1/x^2} = \lim_{x\to\infty} \frac{\frac{-2}{x^2+2x}}{-2/x^3} = \infty$$

$$\lim_{x\to\infty} \ln y$$

So, $\lim_{x\to\infty} \left(1+\frac{2}{x}\right)^{x^2} = \infty \Rightarrow \lim_{n\to\infty} \underline{\left(1+\frac{2}{n}\right)^{n^2}} = \infty \neq 0$

gen. term of the series

So, the given series is divergent.

2d. $\sum_{n=1}^{\infty} \frac{1}{n}$ $\quad \lim_{n\to\infty} \frac{1}{n} = 0$ ✓.

harmonic series diverges.

2e. $\sum_{n=1}^{\infty} \ln\left(\frac{n+2}{n}\right)$ is a divergent telescoping series.

$$\ln\left(\frac{n+2}{n}\right) = \ln(n+2) - \ln(n)$$

HW:

Ratio/root test are inconclusive.

So,

$$\begin{aligned}
a_1 &= \cancel{\ln 3} - \ln 1 \\
a_2 &= \cancel{\ln 4} - \ln 2 \\
a_3 &= \ln 5 - \cancel{\ln 3} \\
&\vdots \\
a_{m-1} &= \cancel{\ln(m+1)} - \cancel{\ln(m-1)} \\
+\quad a_m &= \ln(m+2) - \cancel{\ln(m)} \\
\hline
S_m &= \ln(m+2) + \ln(m+1) - \ln 1 - \ln 2.
\end{aligned}$$

$$\lim_{m\to\infty} S_m = \infty = \sum_{n=1}^{\infty} \ln\left(\frac{n+2}{n}\right).$$

So, the given series diverges.

2f $\sum_{n=1}^{\infty} \frac{1+3^n}{3^{n+5}}$ is divergent.

$$\lim_{n\to\infty} \frac{1+3^n}{3^n \cdot 3^5} = \lim_{n\to\infty} \frac{\cancel{3^n}\left(\frac{1}{3^n}+1\right)}{\cancel{3^n}\cdot 3^5} = \frac{1}{243} \neq 0.$$

$\omega = \frac{1}{x}$

$x\to\infty \therefore \omega \to 0^+$

$\lim_{\omega\to 0^+} \frac{\arcsin\omega}{\omega}$

2g: $\sum_{n=1}^{\infty} n \arcsin\left(\frac{1}{n}\right)$

Replace n by x.

$$\lim_{x\to\infty} x\arcsin\left(\frac{1}{x}\right) = \lim_{x\to\infty} \frac{\arcsin\frac{1}{x}}{\frac{1}{x}} = \lim_{x\to 0^+} \frac{\arcsin x}{x} \overset{\text{H}}{=} \lim_{x\to 0^+} \frac{\frac{1}{\sqrt{1-x^2}}}{1} = 1.$$

So, $\lim_{n\to\infty} n\arcsin\left(\frac{1}{n}\right) = 1 \neq 0$. So, the given series diverges.

2h: $\sum_{n=1}^{\infty} \frac{2^n}{(2n)!} = \frac{2}{2!} + \frac{2^2}{4!} + \frac{2^3}{3!} + \cdots$

Ratio test says that this series is convergent. ← So?

$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\cancel{2^{n+1}}}{(2n+2)!} \cdot \frac{(2n)!}{\cancel{2^n}} = \lim_{n\to\infty} \frac{2}{(2n+1)(2n+2)} = 0 < 1.$

We want to find value of the series.

Later in 9.5

$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots, \quad x \in \mathbb{R}.$

$\frac{1}{2}e^x = \frac{1}{2} + \frac{x}{2} + \frac{x^2}{2 \cdot 2!} + \frac{x^3}{2 \cdot 3!} + \cdots, \quad x \in \mathbb{R}$

$\frac{1}{2}e^{\sqrt{2}} = \frac{1}{2} + \frac{\sqrt{2}}{2} + \frac{2}{2 \cdot 2!} + \frac{2\sqrt{2}}{2 \cdot 3!} + \frac{2 \cdot 2}{2 \cdot 4!} + \cdots$

$\frac{1}{2}e^{-\sqrt{2}} = \frac{1}{2} - \frac{\sqrt{2}}{2} + \frac{2}{2 \cdot 2!} - \frac{2\sqrt{2}}{2 \cdot 3!} + \frac{2 \cdot 2}{2 \cdot 4!} - \cdots$

$+$

$\frac{1}{2}e^{\sqrt{2}} + \frac{1}{2}e^{-\sqrt{2}} = 1 + \underbrace{\frac{2}{2!} + \frac{4}{4!} + \frac{2^3}{6!} + \cdots}_{\text{We want the value of this portion.}}$

So, $\sum_{n=1}^{\infty} \frac{2^n}{(2n)!} = \frac{1}{2}e^{\sqrt{2}} + \frac{1}{2}e^{-\sqrt{2}} - 1.$ //

3. Consider the sequence $\{a_n\}_{n=1}^{\infty}$ given by

$$a_n = \begin{cases} n & \text{if } n \le 100 \\ -\left(\frac{5049}{5050}\right)^{n-101} & \text{if } n > 100 \end{cases}$$

Determine whether or not the infinite series $\sum_{n=1}^{\infty} a_n$ converges. If it converges, find the sum.

Soln:

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \cdots + a_{100} + a_{101} + a_{102} + \cdots$$

$$= 1 + 2 + \cdots + 100 - 1 - \frac{5049}{5050} - \left(\frac{5049}{5050}\right)^2 - \cdots$$

$$= \frac{100.101}{2} - \frac{1}{1 - \frac{5049}{5050}} = 0$$

4: Find a series $\sum_{n=1}^{\infty} a_n$ whose m^{th} partial sum $s_m = 2^{1-m}(2^m - 1)$. Does this series converge?

$S_m = a_1 + a_2 + \dots + a_m$

$-\ S_{m-1} = a_1 + a_2 + \dots + a_{m-1}$

$S_m - S_{m-1} = a_m$

Soln: $S_m = 2^{1-m}(2^m - 1) = 2 - 2^{1-m}$

$a_m = S_m - S_{m-1} = 2 - 2^{1-m} - \underbrace{\left(2 - 2^{1-(m-1)}\right)} = 2^{2-m} - 2^{1-m} = \frac{4}{2^m} - \frac{2}{2^m} = \frac{2}{2^m} = 2^{1-m}$

So, $\sum_{n=1}^{\infty} 2^{1-n} = \sum_{n=1}^{\infty} \frac{1}{2^{n-1}} = \frac{1}{1-\frac{1}{2}} = 2 = \lim_{m \to \infty} S_m = \lim_{m \to \infty} 2 - \underbrace{2^{1-m}}_{\downarrow 0} = 2$

geo. series, $r = \frac{1}{2}$

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