Math 118 9.1

Note Title 3/28/2020

Q1. Determine whether the given sequence is bounded below/above, increasing, decreasing, alternating, convergent, divergent to ∞ or $-\infty$.

(i) $\left\{\frac{n^2-1}{n}\right\}_{n=1}^{\infty}$ (ii) $\left\{\frac{e^n}{\pi^n}\right\}$ (iii) $\left\{\frac{\sin n}{n}\right\}$

$\{a_n\}$ is bdd from above if there is $K>0$ s.th. $a_n \le K$ for all n.

(i) $\left\{\frac{n^2-1}{n}\right\} = \left\{n-\frac{1}{n}\right\}$

- This sequence is <u>not</u> bounded from above

Given any $K>0$ choose $N=K+1$ so that for every $n \geq N = K+1$ we have $n-\frac{1}{n} \geq K$.

$$-\frac{1}{n} \geq \frac{-1}{K+1}$$
$$n-\frac{1}{n} \geq K+1-\frac{1}{K+1} \geq K.$$
$$\underbrace{\qquad}_{\geq 0}$$

- This sequence is bounded from below. by positivity $0 \le n-\frac{1}{n}$ for all $n \geq 1$.

- Let $f(x)=\frac{x^2-1}{x}=x-\frac{1}{x}$. $\Rightarrow$ $f'(x)=1+\frac{1}{x^2}>0$ whenever $x\geq 1$.

So $y=f(x)$ is strictly increasing, $a_n=f(n)$ $\Rightarrow$ $\{a_n\}$ is increasing (eventually) ultimately

- This sequence is neither decreasing nor ultimately decreasing. it's strictly increasing.

compare $\underset{0}{a_1} \le \underset{\frac{3}{2}}{a_2}$

- not alternating. (since $a_2 a_3 \geq 0$)

- not convergent because $\lim_{n\to\infty} n-\frac{1}{n} = \infty$

- it diverges to infinity.

- $\left\{ \frac{e^n}{\pi^n} \right\}_{n=1}^{\infty}$ $\qquad \frac{e}{\pi} \in (0,1)$ $\qquad e \approx 2.7\ldots$, $\pi = 3.141592\ldots$

- The sequence is bdd from $\left| \begin{array}{l} \text{above} \\ \text{below} \end{array} \right|$ as $\frac{e^n}{\pi^n} \leq 1$ for every $n \geq 1$.
as $\frac{e^n}{\pi^n} \geq 0$ for every $n \geq 1$

The seq. is bounded

- This sequence is $\underbrace{\text{neither increasing}}$, nor $\underbrace{\text{ultimately increasing}}$.

$$0.86\ldots = \frac{e}{\pi} \geq \frac{e^2}{\pi^2} = 0.748\ldots$$

Let $f(x) = \left(\frac{e}{\pi}\right)^x \Rightarrow$ is strictly dec.

$\cdot\ f'(x) = \left(\frac{e}{\pi}\right)^x \underbrace{\ln\left(\frac{e}{\pi}\right)}_{<0} < 0$

- It's decreasing, and hence, it's in particular ultimately decreasing.

$f(x) = \left(\frac{e}{\pi}\right)^x$, $x \geq 0$ $\qquad f'(x) < 0 \Rightarrow f$ is a strictly dec. func. $\Rightarrow \{a_n\}$ is stric. decreasing.

- not alternating (because $a_2 a_3 > 0$)

- $\lim\limits_{n \to \infty} \left(\frac{e}{\pi}\right)^n = 0$ $\qquad$, OR, $\qquad$ bdd from below + strictly decreasing $\Downarrow$ it's convergent.

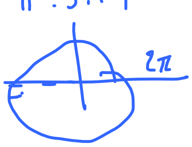
So, $\left\{\left(\frac{e}{\pi}\right)^n\right\}$ is convergent.

- not divergent

- $\left\{ \frac{\sin(n)}{n} \right\} = \left\{ \underbrace{\frac{\sin 1}{1}, \frac{\sin 2}{2}, \frac{\sin 3}{3}}_{\text{positive}}, \underbrace{\frac{\sin 4}{4}, \frac{\sin 5}{5}}_{\text{negative}}, \ldots, \overset{\text{positive}}{\frac{\sin 9}{9}}, \ldots \right\}$

radian sin

$\pi : 3.1415\ldots$



$\sin 30 = -0.988\ldots$

$-1 \leq \sin x \leq 1$

So, $-1 \leq \frac{\sin n}{n} \leq 1$ for all $n \geq 1$, $n \in \mathbb{Z}^+$.

- This seq. is bounded $\left[\begin{array}{l} \text{bdd from above} \\ \text{bdd from below} \end{array} \right]$

- This seq. is neither ultimately increasing nor ultimately decreasing.

- not alternating ($a_1 a_2 > 0$) $\qquad -\frac{1}{n} \leq \frac{\sin n}{n} \leq \frac{1}{n}$.

- $\lim\limits_{n \to \infty} \frac{\sin n}{n} = 0$ by Squeeze thm. $\quad$ The seq. is convergent.

Q2: Find the limit of the sequence $\{a_n\}$ where $a_n = \sin(n\pi)$

Soln: n is a positive integer. So, $\sin(n\pi) = 0$.

So, $\lim\limits_{\substack{n \to \infty \\ n \in \mathbb{Z}^+}} \sin(n\pi) = 0$.

Caution: Wrong $\lim\limits_{x \to \infty} \sin(\pi \underset{\text{real variable}}{x}) = \lim\limits_{x \to \infty} \sin x = DNE$.

$\hat{x} = \pi x$

Q3: Consider the sequence $\{a_n\}$ defined recursively (inductively) as follows:

$$a_1 = 1 \quad \text{and} \quad a_n = 1 + \frac{a_{n-1}}{2} \quad \text{for all } n \geq 2.$$

depth = 1

(i) Show that $a_n < 2$ for all n.

(ii) Show that $a_n < a_{n+1}$ for all n.

(iii) Find the limit of $\{a_n\}$.

$1 = a_1 > 0$

If $a_{n-1} > 0$, $\underbrace{1 + \frac{a_{n-1}}{2}}_{a_n} > 0 \Rightarrow a_n > 0$

(i) $1 = a_1 < 2$

Check if it follows from $a_n < 2$ for some n that $a_{n+1} < 2$. ✓

Indeed, $\underline{a_n < 2} \Rightarrow \frac{a_n}{2} < 1 \Rightarrow \underline{a_{n+1} < 2}$. So, $a_n < 2$ for all n.

$$\underbrace{1 + \frac{a_n}{2}}_{a_{n+1}} < 2$$

(ii) $1 = a_1 < a_2 = 1 + \frac{1}{2} = \frac{3}{2}$

Check if $a_n < a_{n+1}$ for some n $\Rightarrow a_{n+1} < a_{n+2}$. Yes. So, $a_n < a_{n+1}$ for all n. ✓

$$\underbrace{1 + \frac{a_n}{2}}_{a_{n+1}} < \underbrace{\frac{a_{n+1}}{2} + 1}_{a_{n+2}}$$

Tech #2: $f(x) = 1 + \frac{x}{2}$ is inc. func.

$f'(x) = \frac{1}{2}$.

$a_n < a_{n+1}$. Take f of both sides.

$$\underline{a_{n+1}} = f(a_n) < f(a_{n+1}) = \underline{a_{n+2}}$$

(iii) The sequence is bounded from above and it's stric. increasing.

So it does have a limit.

Say $L = \lim_{n \to \infty} a_n$. Take $\lim_{n \to \infty}$ of both sides of recursive formula.

$$\lim_{n \to \infty} a_n = \lim_{n \to \infty} 1 + \frac{a_{n-1}}{2} = 1 + \frac{1}{2} \lim_{n \to \infty} a_{n-1}$$

$$L = 1 + \frac{1}{2} \cdot L \Rightarrow \boxed{L = 2}$$

Q4: If exists, find the limit of the following sequences.

- $\{n-\sqrt{n^2-4n}\}$

$$\lim_{n\to\infty} n-\sqrt{n^2-4n} = \lim_{n\to\infty} \frac{n^2-(n^2-4n)}{n+\sqrt{n^2-4n}} = \lim_{n\to\infty} \frac{4n}{n+\sqrt{n^2-4n}}$$

$$\sqrt{n^2\left(1-\frac{4}{n}\right)} = n\sqrt{1-\frac{4}{n}}$$

$$= \lim_{n\to\infty} \frac{4\cancel{n}}{\cancel{n}\left(1+\sqrt{1-\frac{4}{n}}\right)} = \frac{4}{2} = \underline{\underline{2}}$$

- $\{n \sin\frac{1}{n}\}$

Use the x variable

$$w=\frac{1}{x} \qquad x\to\infty, \quad w\to 0^+$$

$$\lim_{x\to\infty} x\sin\frac{1}{x} = \lim_{w\to 0^+} \frac{\sin w}{w} =$$

$$\lim_{x\to\infty} x\sin\frac{1}{x} = \lim_{x\to 0^+} \frac{\sin x}{x} = 1 \quad \text{So,} \quad \lim_{n\to\infty} n\sin\frac{1}{n} = 1.$$

• $\left\{ \dfrac{n!}{n^n} \right\}$

$$0 \le a_n = \frac{1 \times 2 \times 3 \times \cdots \times n}{n \times n \times n \cdots \times n} \le \frac{1}{n}$$

$\le 1 \quad \le 1$

Take $\lim_{n\to\infty}$ of all sides.

$$0 \le \lim_{n\to\infty} a_n \le \lim_{n\to\infty} \frac{1}{n} = 0 \Rightarrow \lim_{n\to\infty} a_n = 0.$$

$$e = \lim \left(1 + \frac{1}{n}\right)^n$$

Remark: This sequence is eventually decreasing because

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!} = \frac{n^n}{(n+1)^n} = \left(\frac{n}{n+1}\right)^n,$$

if $n >> 0$ then $\dfrac{a_{n+1}}{a_n} \approx \dfrac{1}{e}$

• $\left\{ \dfrac{\cos(4n)}{n+1} \right\}$ ~ not eventually decreasing.

$$-\frac{1}{n+1} \le \frac{\cos 4n}{n+1} \le \frac{1}{n+1}$$

Take $\lim_{n\to\infty}$ of all sides. $\quad \lim_{n\to\infty} \dfrac{\cos 4n}{n+1} = 0$

• $\left\{ \dfrac{n^2}{n^3+1} \right\}$

$$\lim_{n\to\infty} \frac{n^2}{n^3+1} = \lim_{n\to\infty} \frac{\cancel{n^2}}{\underset{n}{\cancel{n^3}}\left(1+\frac{1}{n^3}\right)} = 0$$

• $\left\{ \dfrac{n^2 2^n}{n!} \right\}$ $\qquad (n+1)! = (n+1)\underbrace{n!}_{\geq 3^n} \geq 3^{n+1}$

If $n \geq 7$ then $n! \geq 3^n$. So, $0 \leq \dfrac{n^2 2^n}{n!} \leq \dfrac{n^2 2^n}{3^n} \Rightarrow \underset{0}{\underset{\|}{0}} \leq \underset{0}{\underset{\|}{\lim_{n\to\infty} \frac{n^2 2^n}{n!}}} \leq \underset{0}{\underset{\|}{\lim_{n\to\infty} \frac{n^2 2^n}{3^n}}}$

Compute $\lim_{n\to\infty} \dfrac{n^2 2^n}{3^n}$.

$$\lim_{x\to\infty} \frac{x^2 2^x}{3^x} = \lim_{x\to\infty} \frac{x^2}{\left(\frac{3}{2}\right)^x} \overset{H}{=} \lim_{x\to\infty} \frac{2x}{\left(\frac{3}{2}\right)^x \ln\frac{3}{2}} \overset{H}{=} \lim_{x\to\infty} \frac{2}{\left(\frac{3}{2}\right)^x \left(\ln\frac{3}{2}\right)^2} = 0$$

$$\Rightarrow \lim_{n\to\infty} \frac{n^2 2^n}{3^n} = 0.$$

Q5: Let $\{a_n\}$ be the sequence defined ~~recursively~~ inductively as follows:

$a_1 = 1, \quad a_2 = 2 \quad$ and $\quad a_n = a_{n-1} + a_{n-2}$ for all $n \geq 3$. $\qquad a_n > 1$

depth = 2

You are given that the sequence $\left\{ \frac{a_{n+1}}{a_n} \right\}$ has a positive limit.

Find the limit $\lim_{n \to \infty} \frac{a_{n+1}}{a_n}$

$$L = \lim_{n \to \infty} \frac{a_{n+1}}{a_n}$$

$$L = \lim_{n \to \infty} \frac{a_n}{a_{n-1}}$$

$$\frac{a_{n+1}}{a_n} = \frac{a_n + a_{n-1}}{a_n} = 1 + \frac{a_{n-1}}{a_n}$$

Take limit of all sides

unless $L = 0$

$$\underbrace{\lim_{n \to \infty} \frac{a_{n+1}}{a_n}}_{L} = \lim_{n \to \infty} \left(1 + \frac{a_{n-1}}{a_n} \right) = 1 + \underbrace{\lim_{n \to \infty} \frac{a_{n-1}}{a_n}}_{}.$$

$$L = 1 + \frac{1}{L} \Rightarrow L^2 - L - 1 = 0$$

$$\Delta = 1 + 4 = 5$$

$$L = \frac{1 \mp \sqrt{5}}{2}$$

$$L = \frac{1 + \sqrt{5}}{2}$$