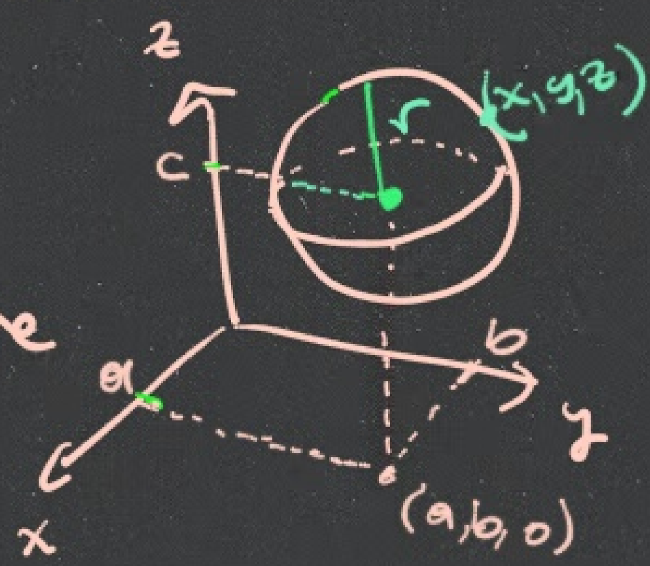
10.5. Quadric Surfaces

paraboloid hyperboloid

- We have seen that equations of the form

$$Ax+By+Cz=D \qquad \text{or } Ax+By+Cz-D=0$$

describe a plane. This is a degree 1 equation in x,y,z.

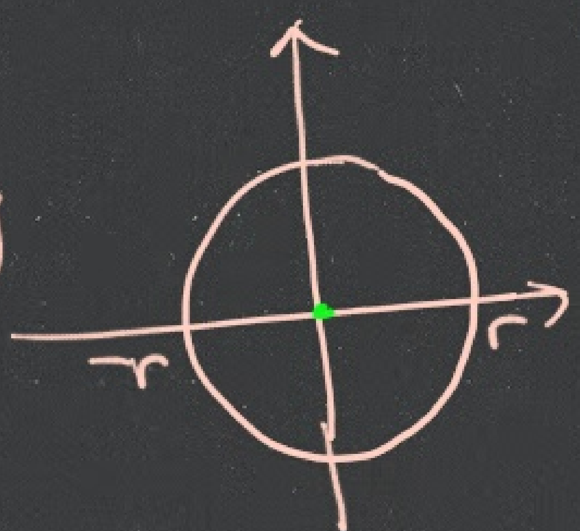


- We mentioned briefly that $(x-a)^2+(y-b)^2+(z-c)^2=r^2$ is a sphere.
This is a degree 2 equation in x,y,z.

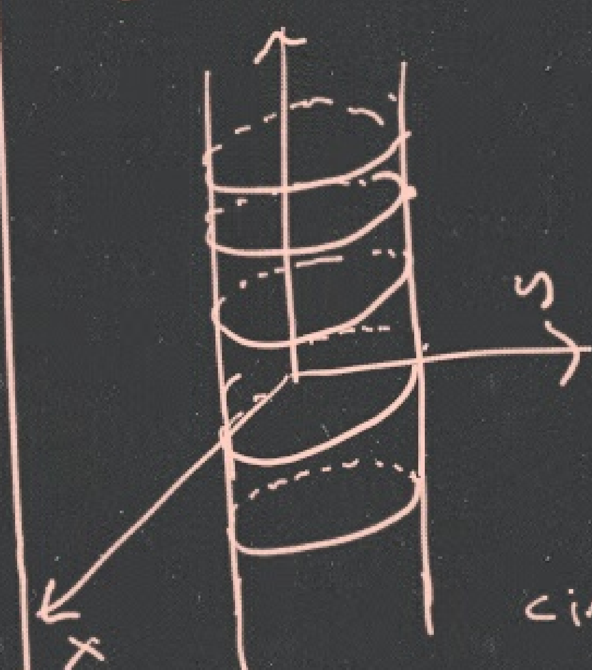
Example: $x^2+y^2=r^2$ or $x^2+y^2-r^2=0$ a degree 2 eqn.

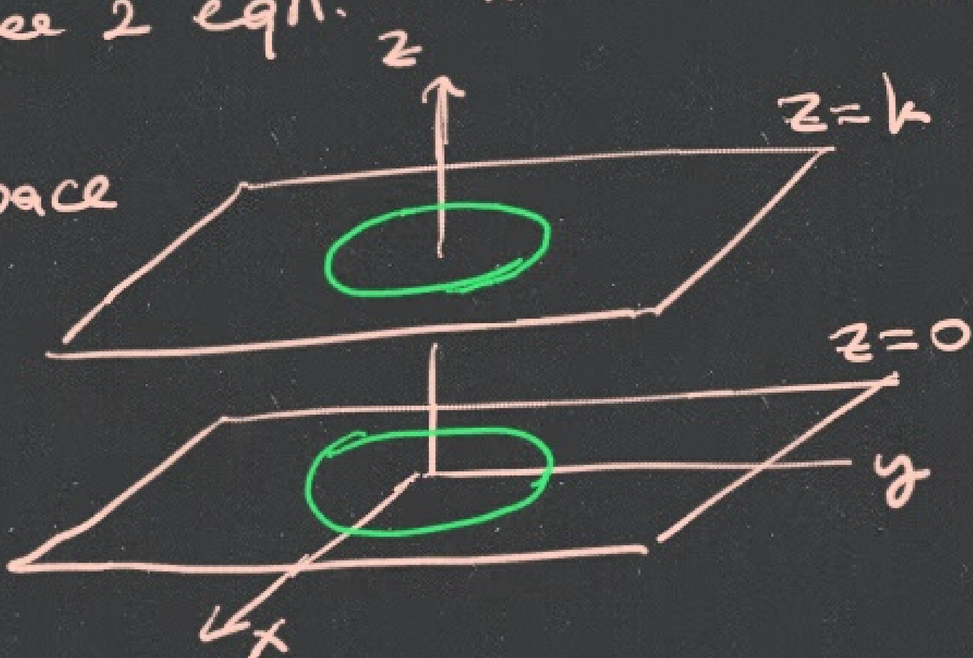
its graph in 2-space is the circle

$$\{(x,y)\mid x^2+y^2=r^2\}$$



its graph in 3-space is the cylinder





$\{(x,y,z)\mid x^2+y^2=r^2\}$

no z in the equation

circular cylinder

The most general second degree equation in x,y,z is

$$(*)\ Ax^2+By^2+Cz^2+Dxy+Exz+Fyz+Gx+Hy+Iz+J=0,$$ not all of A,B,C,D,E,F are 0.

The equation $(*)$ can sometimes be factored as:

$$(A_1x+B_1y+C_1z-D_1)(A_2x+B_2y+C_2z-D_2)=0$$

. What will be its graph?

Example: $(x+y+z-3)\cdot(z-3)=xz+yz+z^2-z-3x-3y-3z+3=$

$$=z^2+xz+yz-3x-3y-4z+3=0 \quad (*)$$

$$\Leftrightarrow \underbrace{x+y+z-3=0}_{\mathcal{P}_1} \text{ or } \underbrace{z-3=0}_{\mathcal{P}_2} \Rightarrow (*) \text{ has graph } \mathcal{P}_1\cup\mathcal{P}_2$$

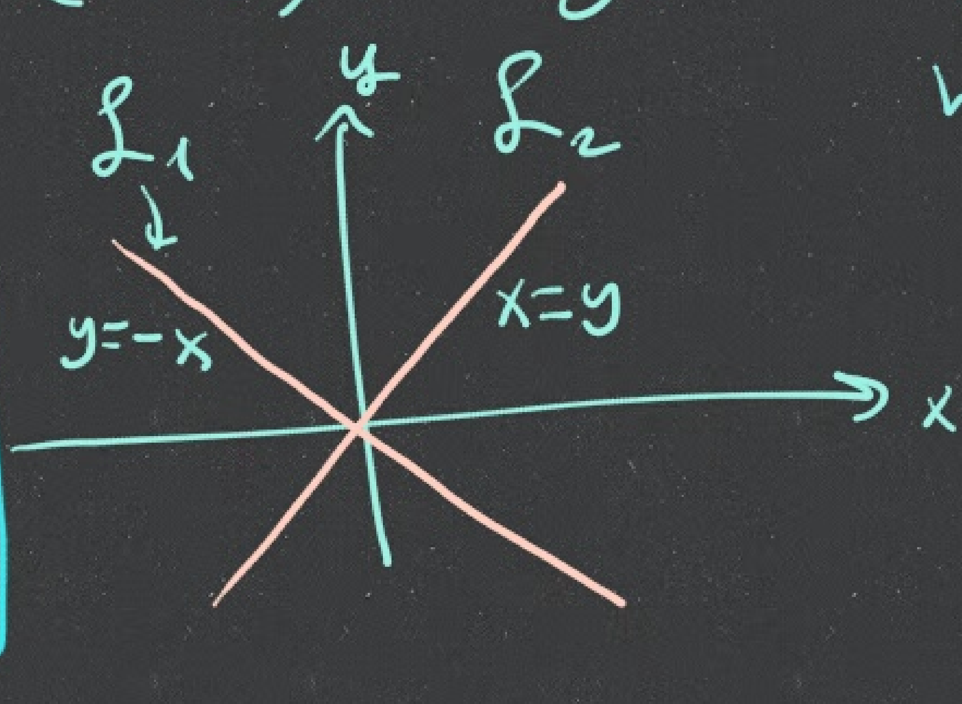
Remark: If $A_2=A_1$, $B_2=B_1$, $C_2=C_1$, $D_2=D_1$ there is only one plane

Example:

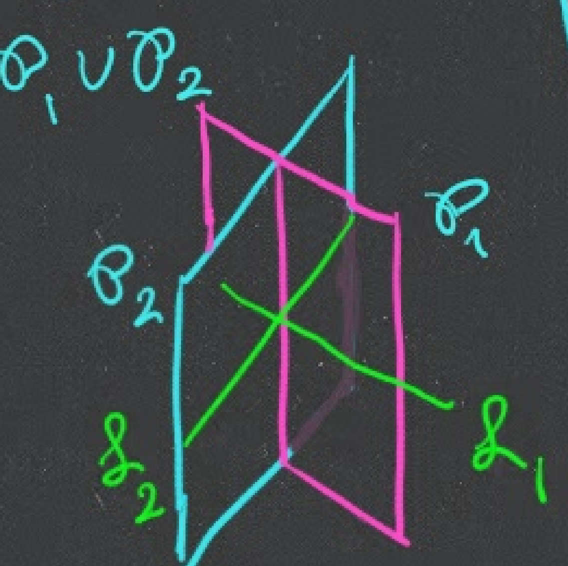
In $\mathbb{R}^2$,

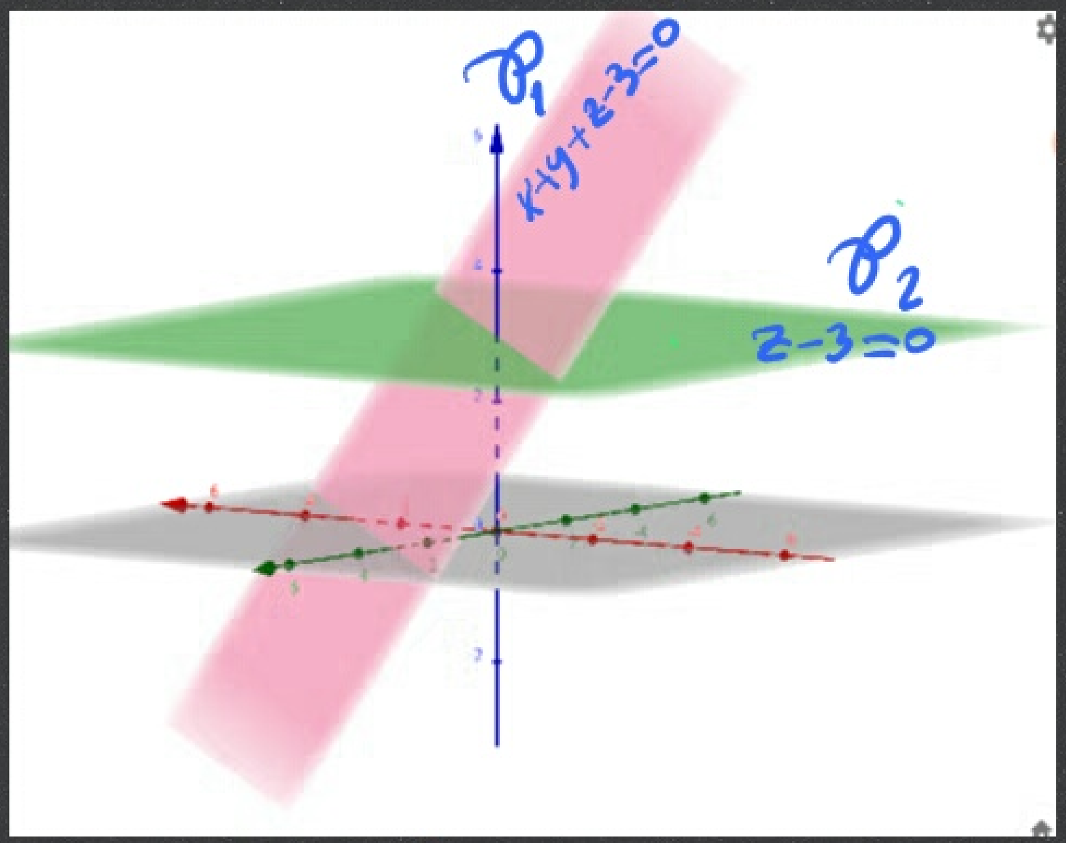
$$\underbrace{(x+y)}_{\overset{0}{\scriptstyle \mathcal{L}_1}}\cdot\underbrace{(x-y)}_{\overset{0}{\scriptstyle \mathcal{L}_2}}=x^2-y^2=0$$

has graph $\mathcal{L}_1\cup\mathcal{L}_2$



In $\mathbb{R}^3$, its graph $\mathcal{P}_1\cup\mathcal{P}_2$





Geoalgebra

When there is no factorization, the surface defined by the equation

(*) $Ax^2+By^2+Cz^2+Dxy+Exz+Fyz+Gx+Hy+Iz+J=0$

is not going to be flat like a plane, it may contain straight lines.

the surface is called a quadric surface.

For simplicity, we will assume $D=E=F=0$ (they rotate the surfaces)

Cylinders are the easiest surfaces; when there are only two variables in the equation, the surface it describes consists of all lines parallel to a given line which pass through a given plane curve.

Circular cylinders $x^2+y^2=1$ or $y^2+z^2=1$, ...

$(x^2+y^2-1=0)$ $(y^2+z^2-1=0)$

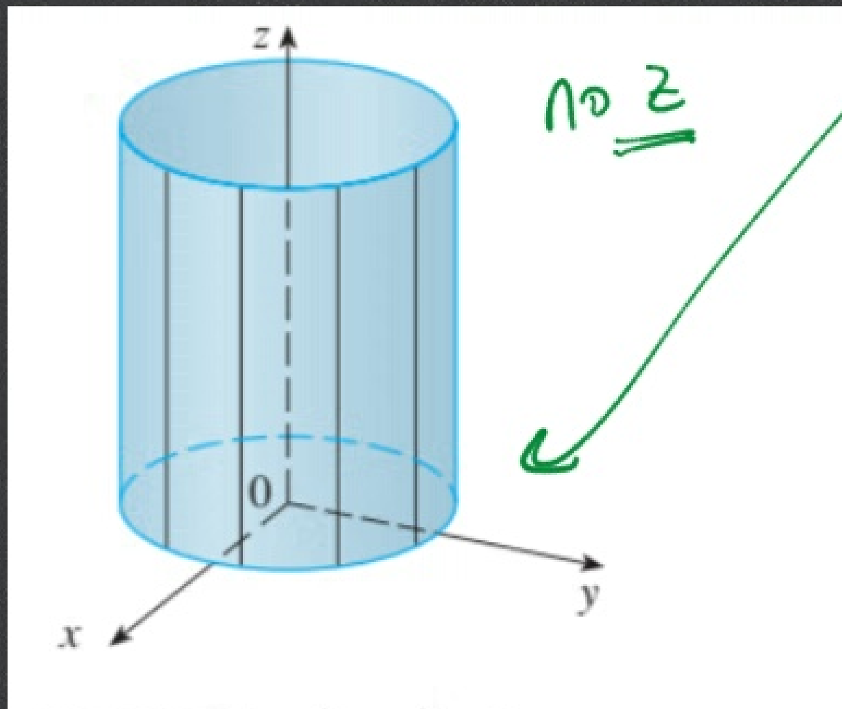


FIGURE 2 $x^2+y^2=1$

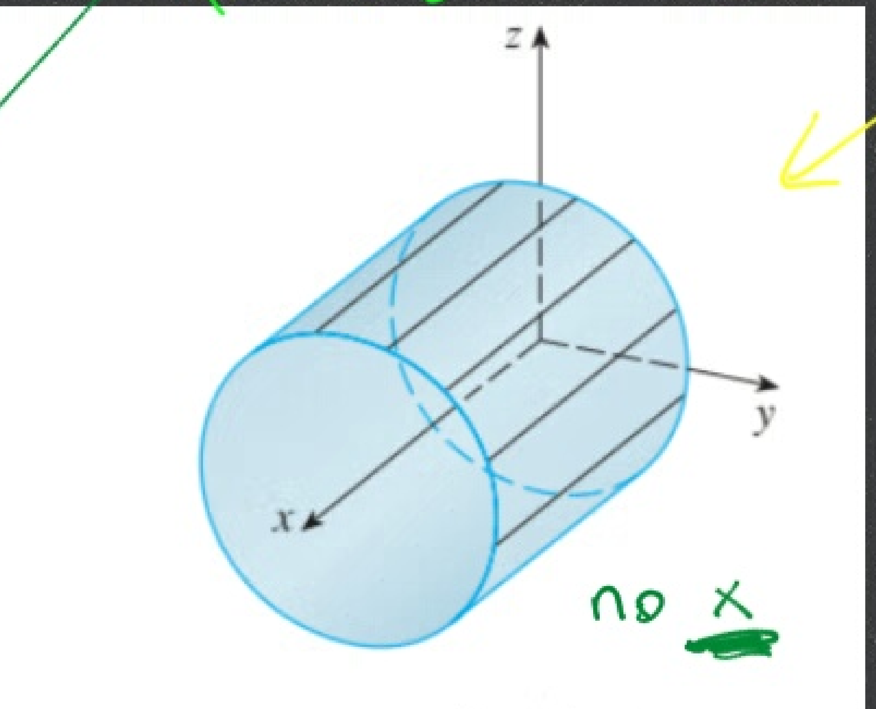
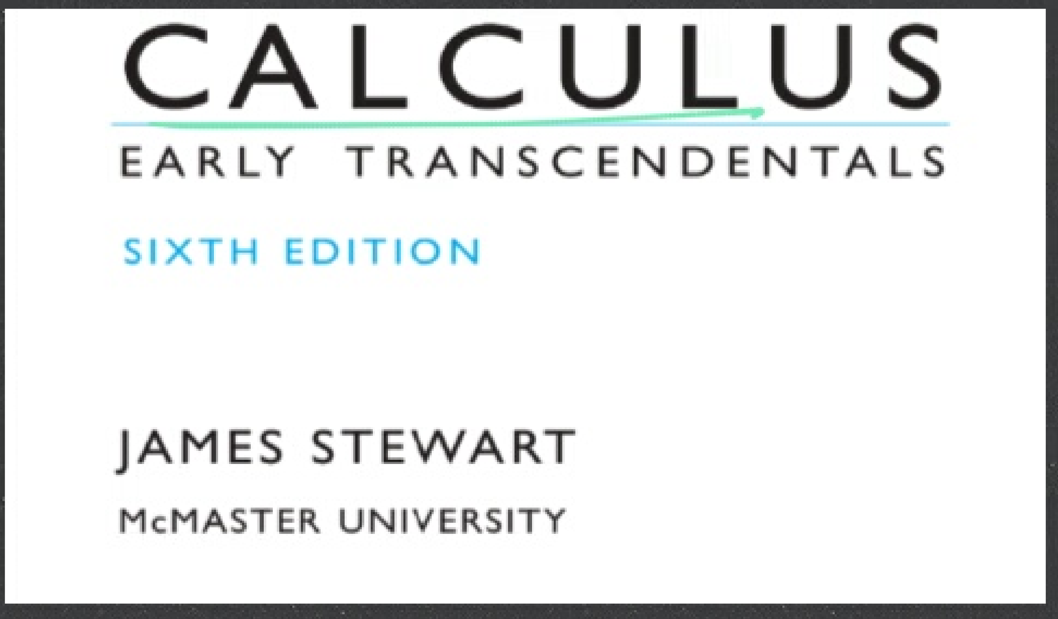


FIGURE 3 $y^2+z^2=1$

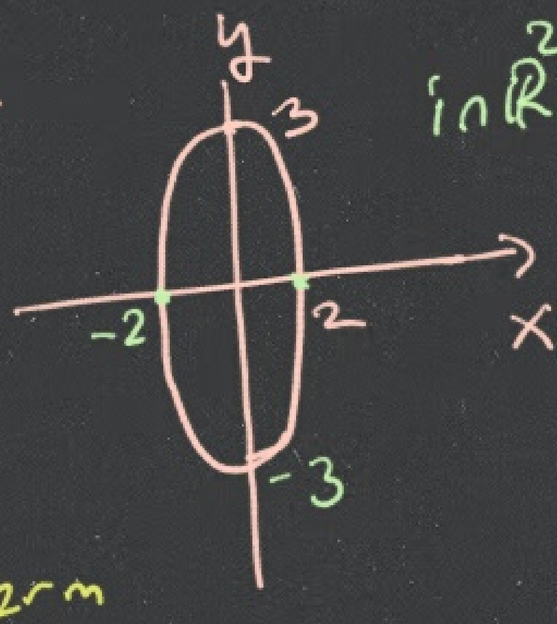
The Pictures are from:



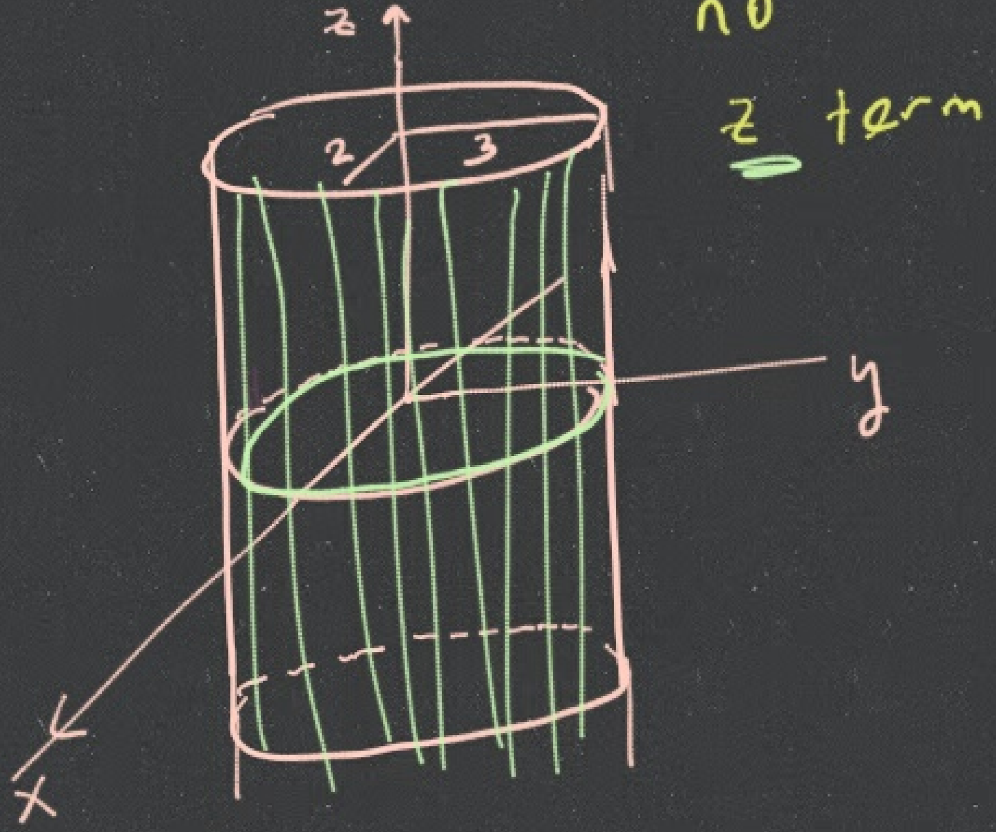
or by Geoalgebra

Elliptic cylinder

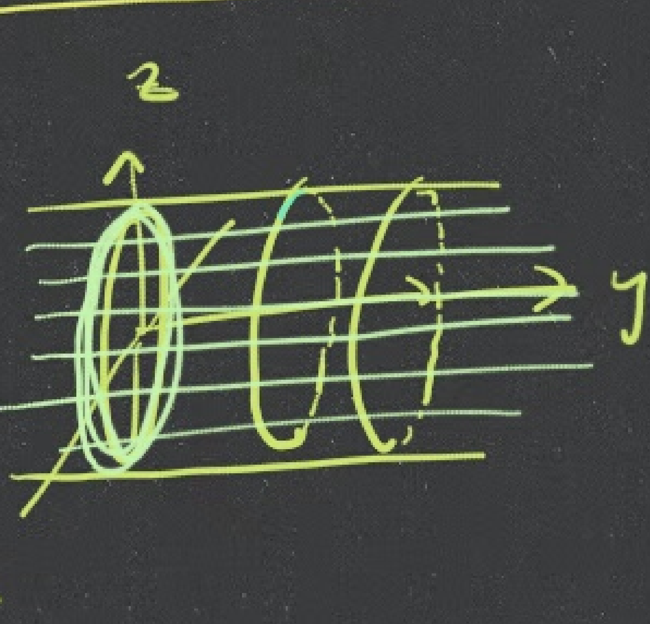
ellipse in $\frac{x^2}{2^2}+\frac{y^2}{3^2}=1$ in $\mathbb{R}^2$



no z term

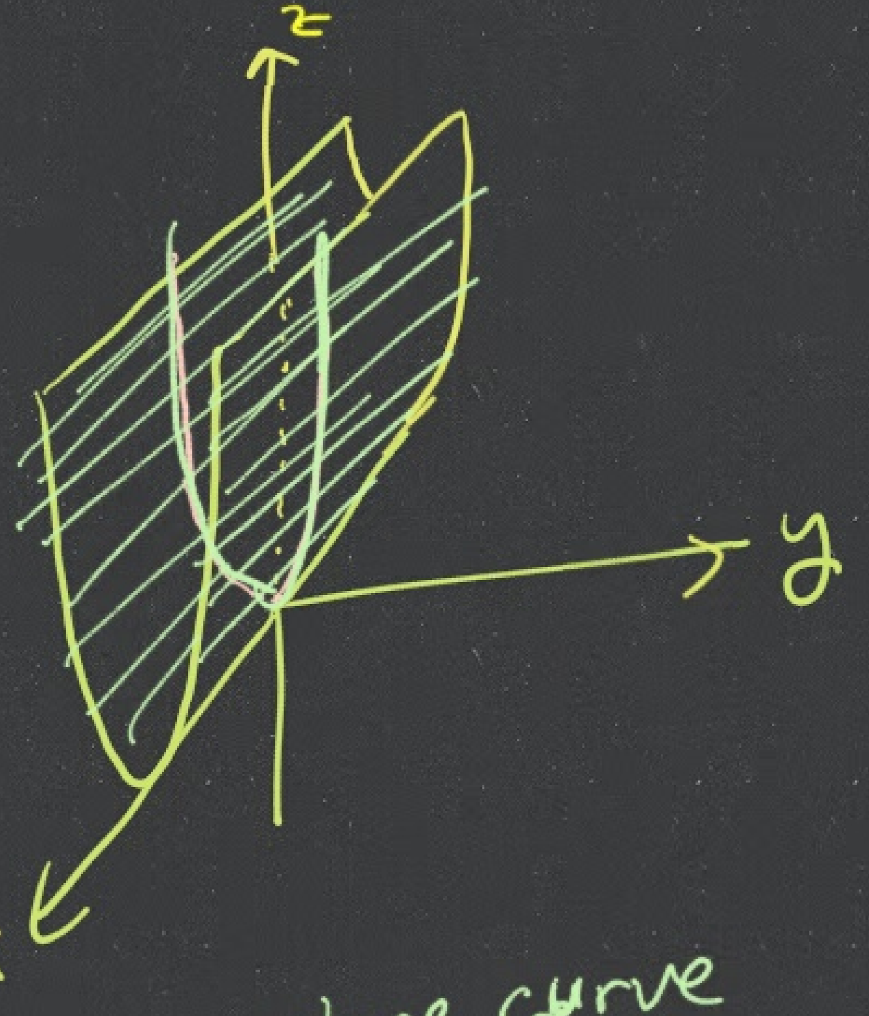


Elliptic cylinder



$\frac{x^2}{2^2}+\frac{z^2}{3^2}=1$ no y-term

Parabolic Cylinder



$z-y^2=0$, $z=y^2$

no x term

xz-plane curve

$z=x^2$

no y term.

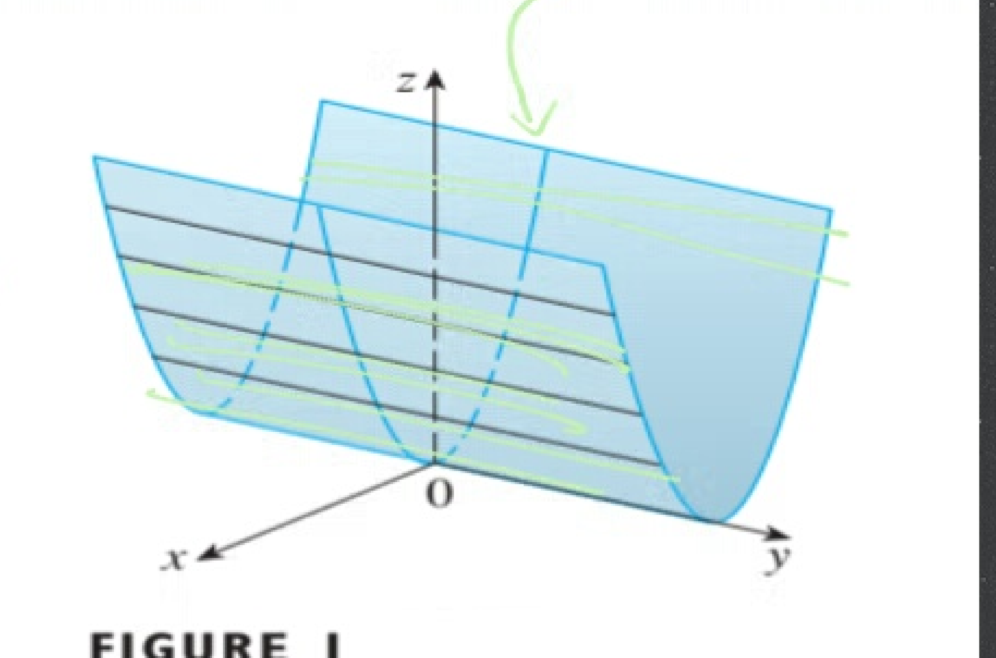
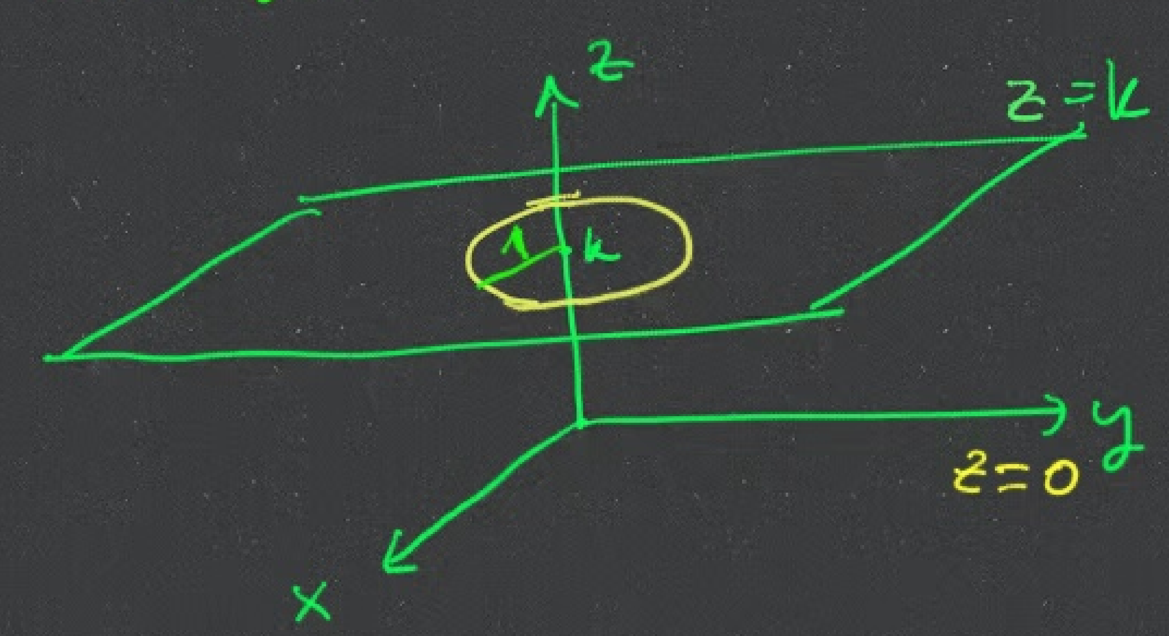


FIGURE 1
The surface $z=x^2$ is a parabolic cylinder.

Remark

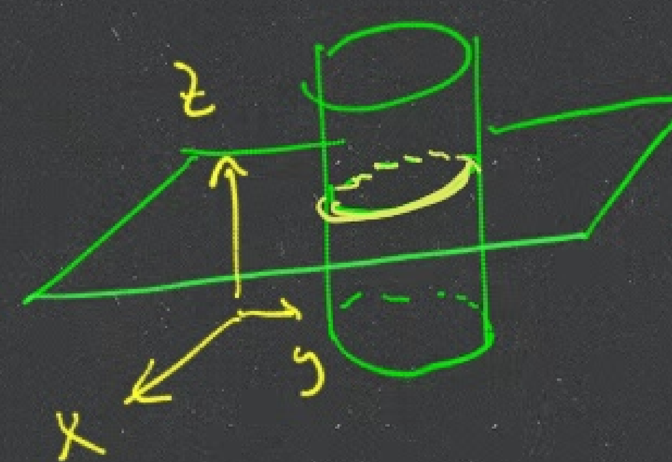
The yellow circle below is a circle of radius 1, centered at $(0,0,k)$ and is parallel to xy-plane.



Does it have a single equation with x, y, z?

No, it can be expressed as the intersection of a pair of surfaces. We need two eqns. together.

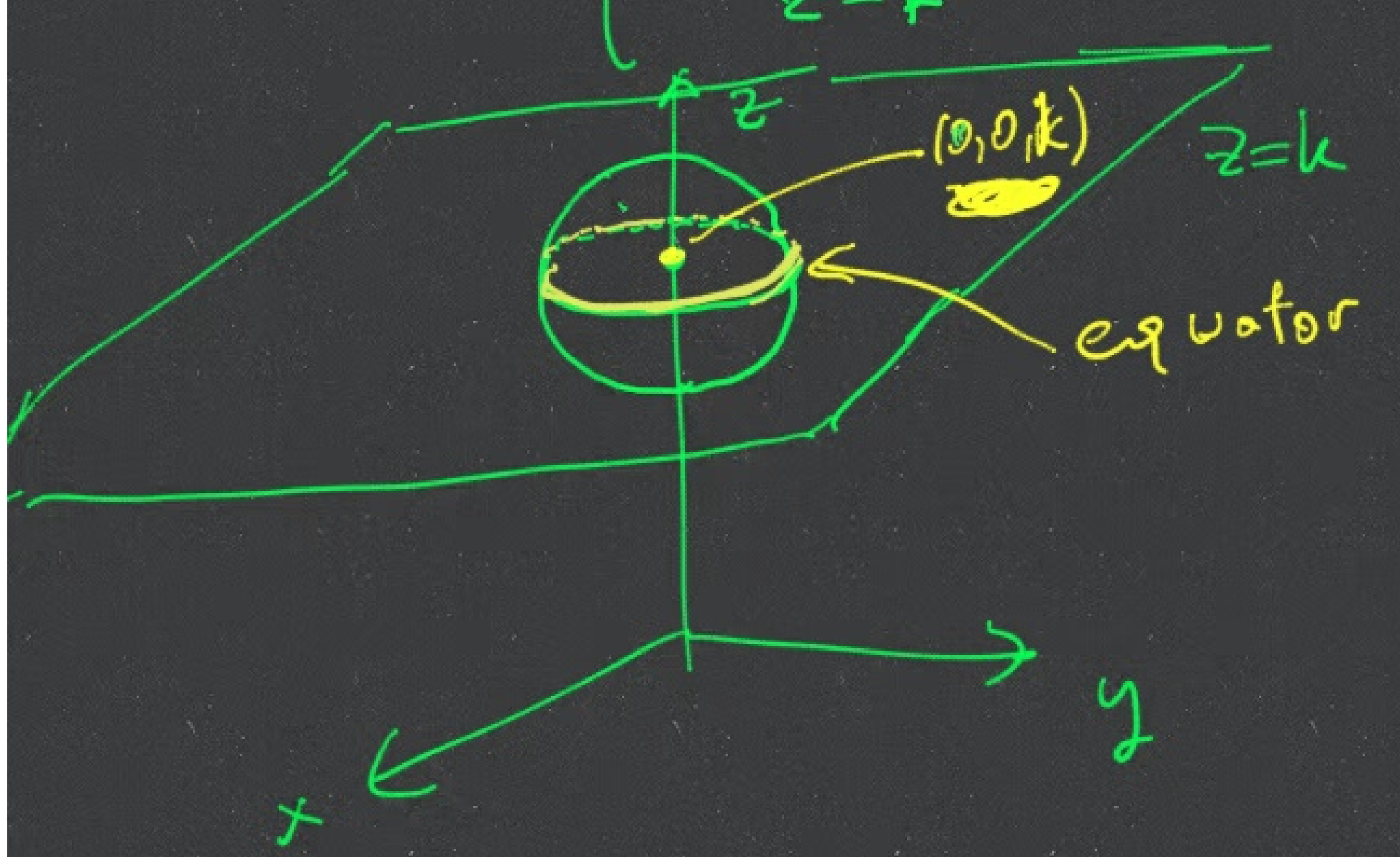
(1) cylinder and plane



$$\begin{cases} x^2+y^2=1 & \leftarrow \text{a cylinder in } \mathbb{R}^3\text{, a circle in xy-plane} \\ z=k & \leftarrow \text{a plane in } \mathbb{R}^3 \text{ parallel to xy-plane} \end{cases}$$

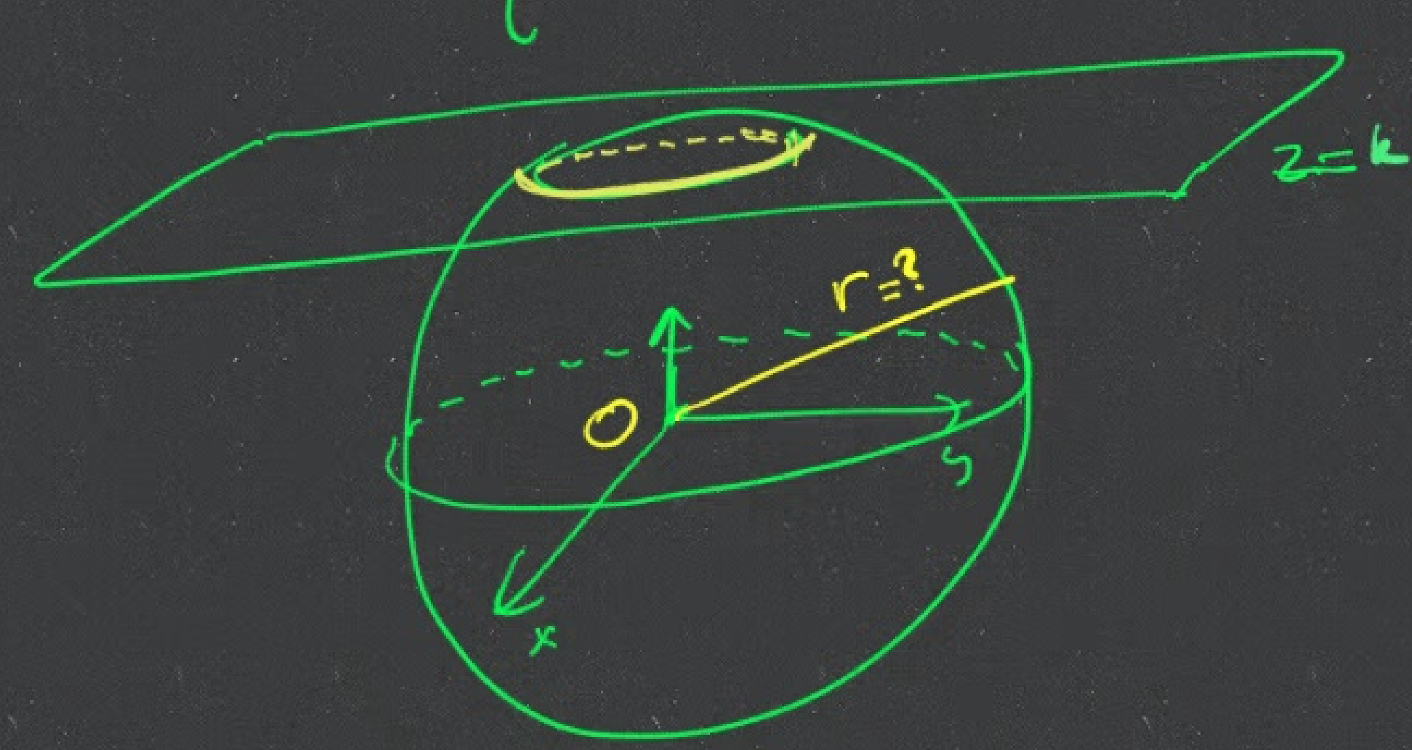
(2) sphere and plane

$$\begin{cases} x^2+y^2+(z-k)^2=1 \\ z=k \end{cases}$$



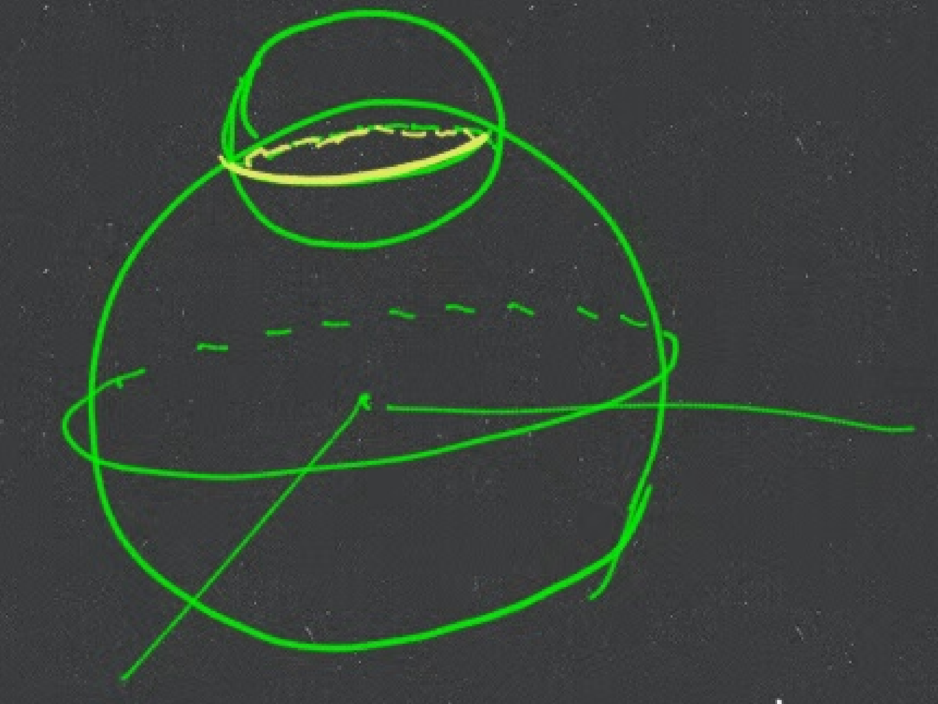
(3) sphere & plane

$$\begin{cases} x^2+y^2+z^2=k^2+1=\left(\underbrace{\sqrt{k^2+1}}_{r}\right)^2 \\ z=k \end{cases}$$



(4) two spheres

$$\begin{cases} x^2+y^2+z^2=k^2+1 \\ x^2+y^2+(z-k)^2=1 \end{cases}$$

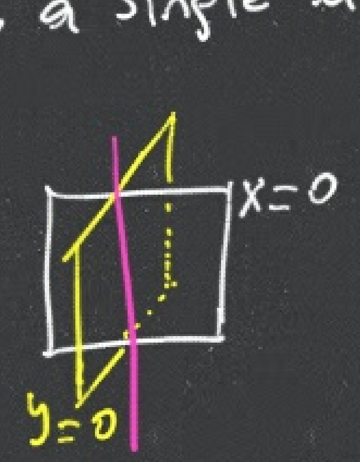


Similarly, there is no single equation with x, y, z to describe a line.

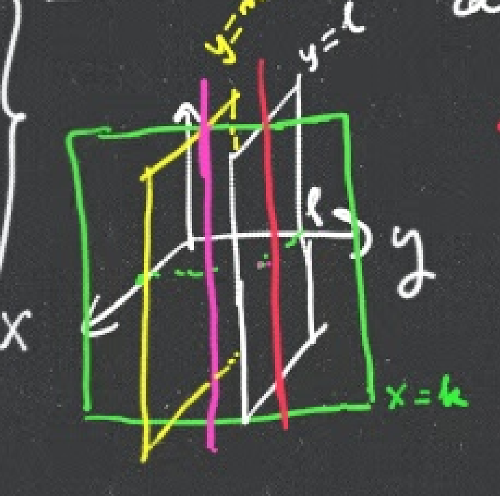
Recall that intersection of two nonparallel planes is a line.

Example: $\begin{cases} x=0 \\ y=0 \end{cases}$ two planes determine a single line,

$\{(0,0,z) \mid z\in\mathbb{R}\}$ pink line



Example $\begin{cases} x=k \\ y=l \\ y=m \end{cases}$ three planes determine a pair of lines



$L_1=\{(k,l,z) \mid z\in\mathbb{R}\}$ red line

$L_2=\{(k,m,z) \mid z\in\mathbb{R}\}$ pink line

- As a parametric equation the above yellow circle centered at $(0,0,k)$ of radius 1 can be given by $f(\theta)=(\cos(\theta), \sin(\theta), k)$, $\theta\in\mathbb{R}$
- As a parametric equation a line can be given by $L(t)=(x_0+ta, y_0+tb, z_0+tc)$, $t\in\mathbb{R}$.

CROSS – SECTIONS / TRACES of a Surface.

To visualize graphs of equations in 3-space it is useful to look at its intersections with planes parallel to the coordinate planes, xy, xz, yz planes.

(horizontal) planes parallel to xy-plane have equation $z=k$, for $k\in\mathbb{R}$

(vertical) " " " xz-plane " " $y=k$, "

(vertical) " " " yz-plane $x=k$, "

Intersections of a surface $S: f(x,y,z)=0$ with such planes are called cross-sections or traces of S.

$y=k$ and $x=k$ give vertical traces

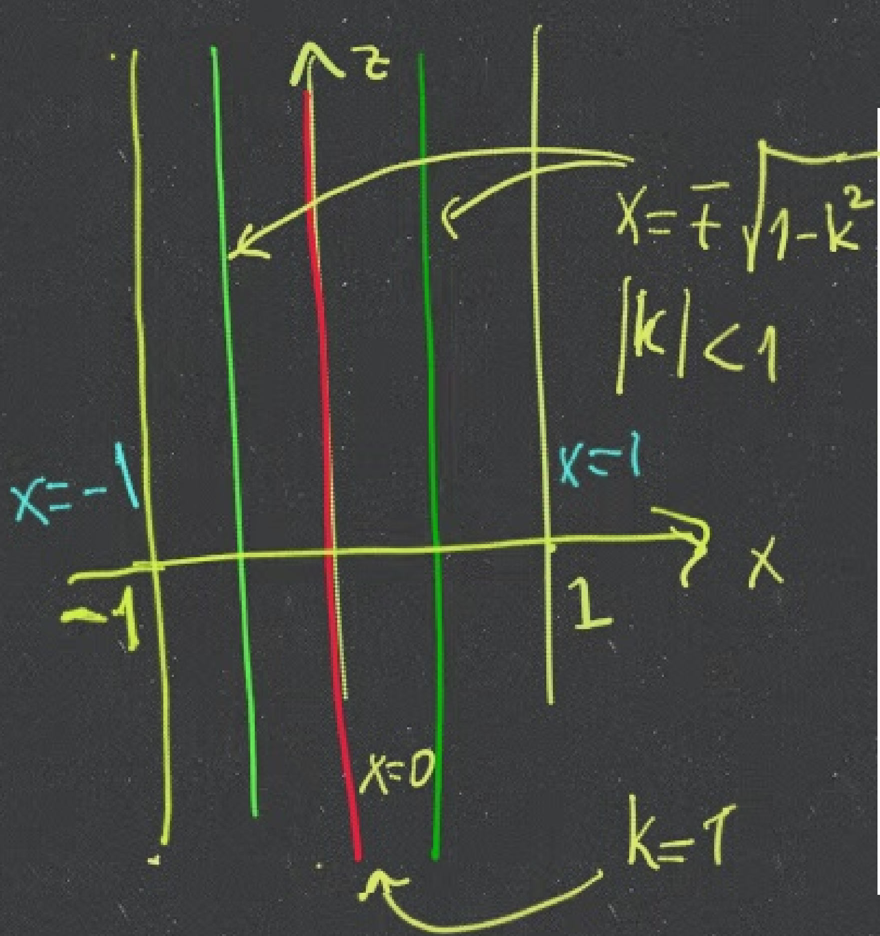
$z=k$ gives horizontal traces, for $k\in\mathbb{R}$.

Example What are the vertical and horizontal traces (cross-sections) of $x^2+y^2=1$?

There is no z in the equation $x^2+y^2=1$ so all horizontal traces are circles.

For vertical traces: consider $y=k$'s first. For $y=k$, $x^2+y^2=x^2+k^2=1$

$$\Rightarrow x=\mp\sqrt{1-k^2} \quad \text{for } |k|\le 1.$$



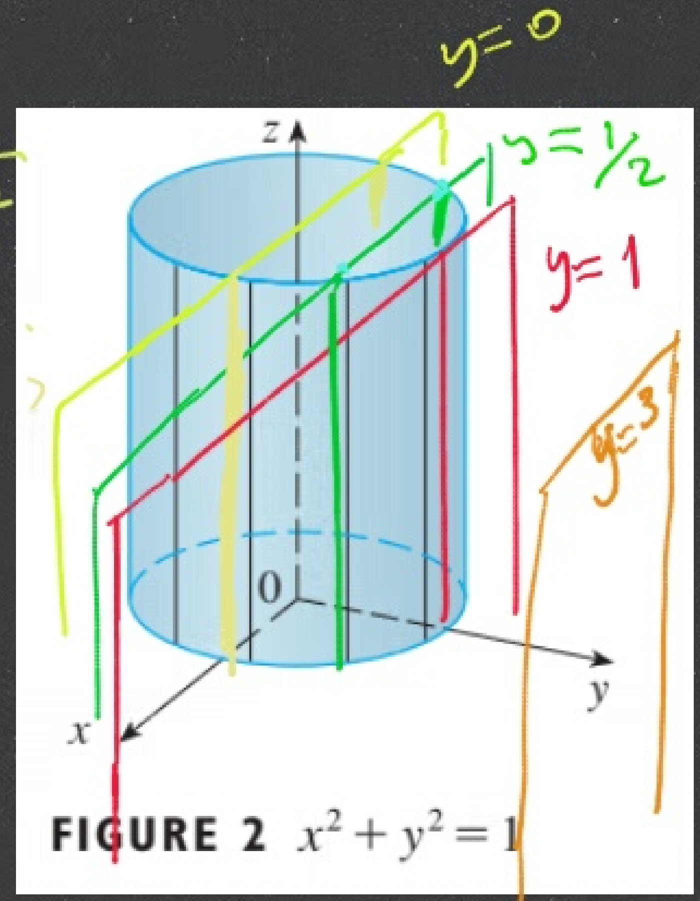


FIGURE 2 $x^2+y^2=1$

If $k=1$, the plane $y=1$, intersects with the cylinder in one line, $\begin{Bmatrix} x=0 \\ y=1 \end{Bmatrix}$ ⟵ red line

If $|k|<1$, the plane $y=k$ intersects with the cylinder in two lines $\begin{cases} x=\sqrt{1-k^2} \\ x=-\sqrt{1-k^2} \\ y=k \end{cases}$

Similarly for $x=k$; $y=\mp\sqrt{1-k^2}$; so the traces are the lines

$\begin{cases} y=\sqrt{1-k^2} \\ y=-\sqrt{1-k^2} \\ x=k \end{cases}$ one line if $k=1$, two lines if $|k|<1$

Therefore all vertical traces are a single line or a pair of lines.

Example: Find all traces of the parabolic cylinder $z=x^2$.

For any k, $y=k$ gives $z=x^2$ (no y term) so it is the same parabola so vertical traces on $y=k$ are all parabolas.

$$\downarrow \quad \begin{Bmatrix} y=k \\ z=x^2 \end{Bmatrix}$$

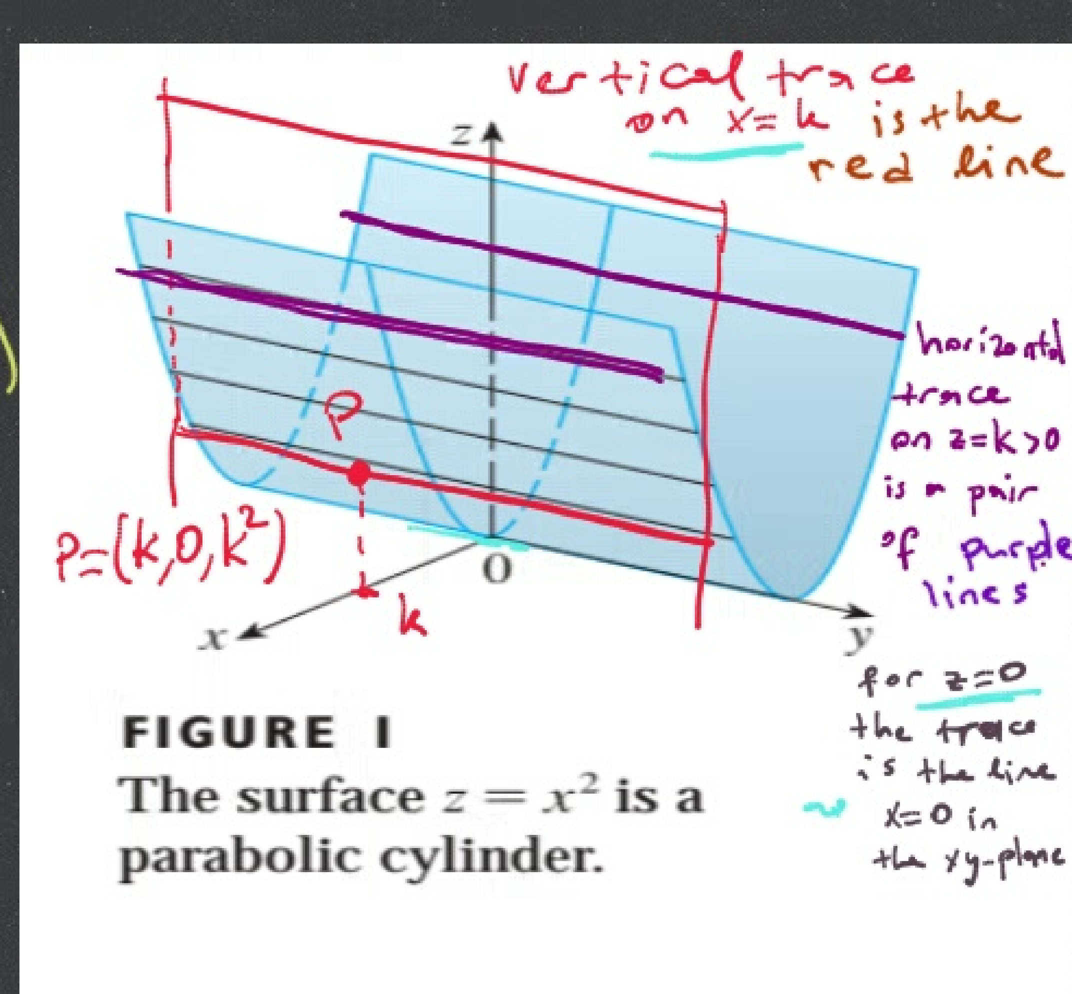
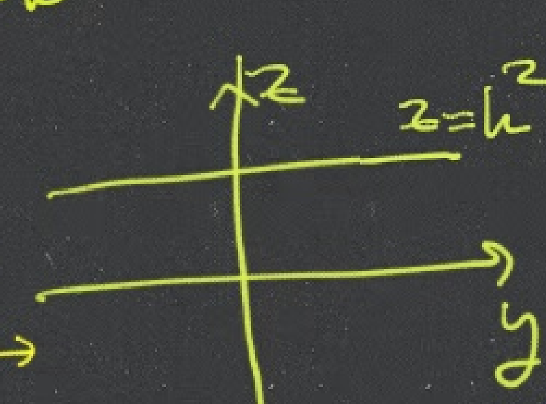


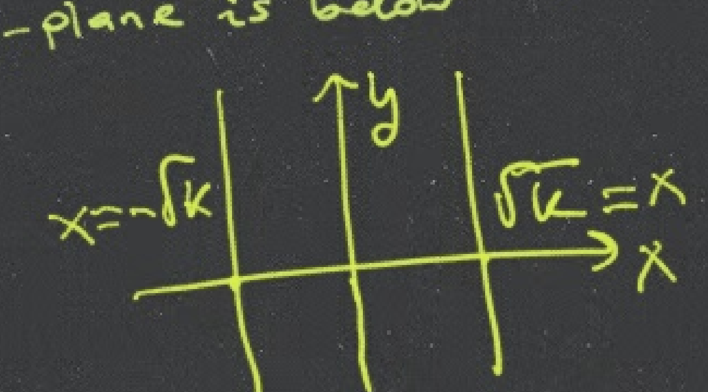
FIGURE 1 The surface $z=x^2$ is a parabolic cylinder.

for $x=k \Rightarrow z=x^2=k^2$ is the red line $\begin{cases} x=k \\ z=k^2 \end{cases}$ whose projection to yz-plane is →

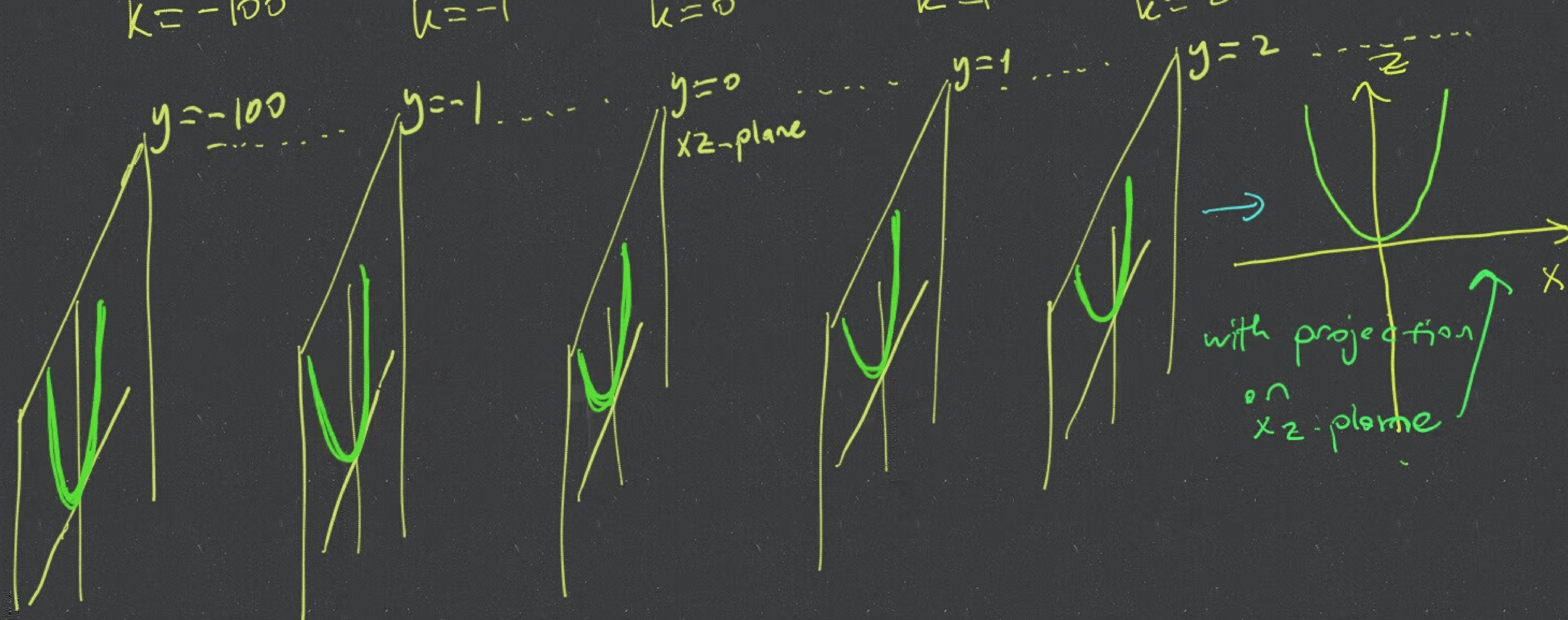


For $z=k>0 \Rightarrow k=x^2 \Rightarrow x=\mp\sqrt{k}$

$\begin{cases} z=k \\ x=\sqrt{k} \\ x=-\sqrt{k} \end{cases}$ describes a single line if $k=0$, " a pair of lines if $k\neq 0$, purple lines in the picture whose projection onto the xy-plane is below



$k=-100$ $k=-1$ $k=0$ $k=1$ $k=2$



Six types of QUADRIC SURFACES

Remember that the equation for a quadric surface is of the form

$Ax^2+By^2+Cz^2+Gx+Hy+Iz+J=0$, in some books it is written as $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}+\cdots=0$; in the following pictures this is this case.

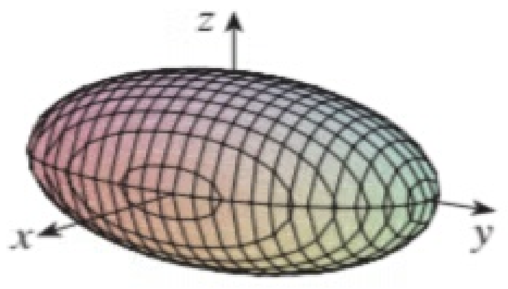
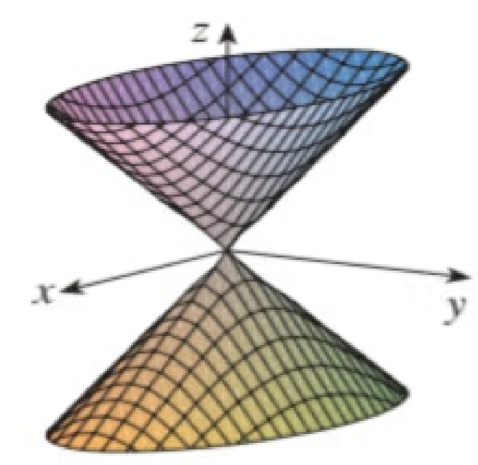
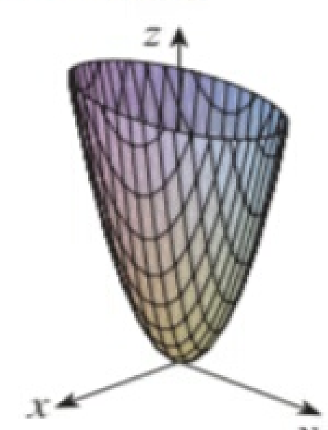
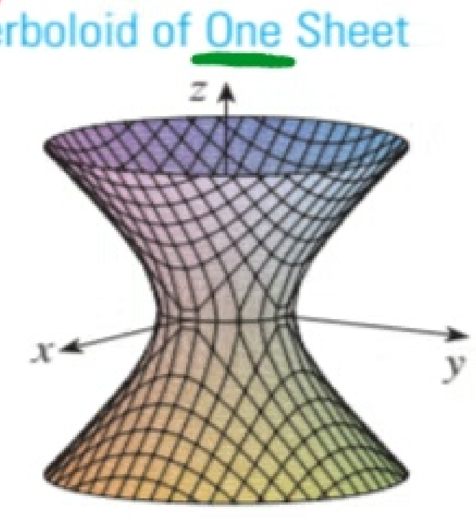
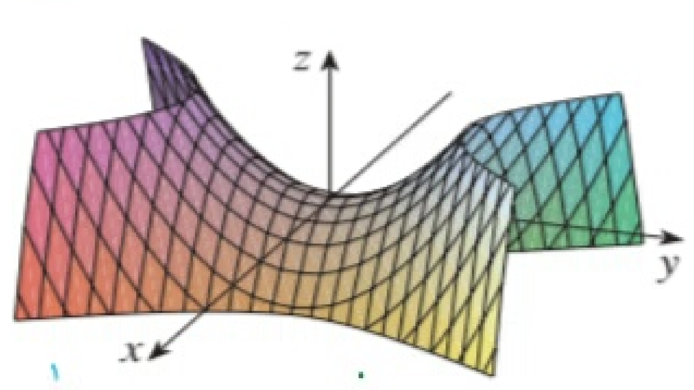
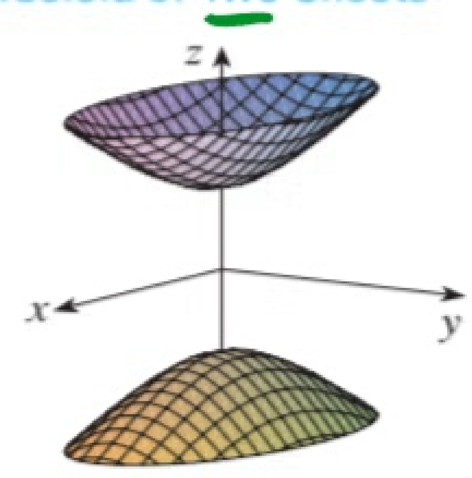
The six types of quadric surfaces listed in our book are:

① spheres, ② cylinders, ③ ellipsoids, ④ cones, ⑤ paraboloids (elliptic, hyperbolic), ⑥ hyperboloids (one sheet, two sheets)

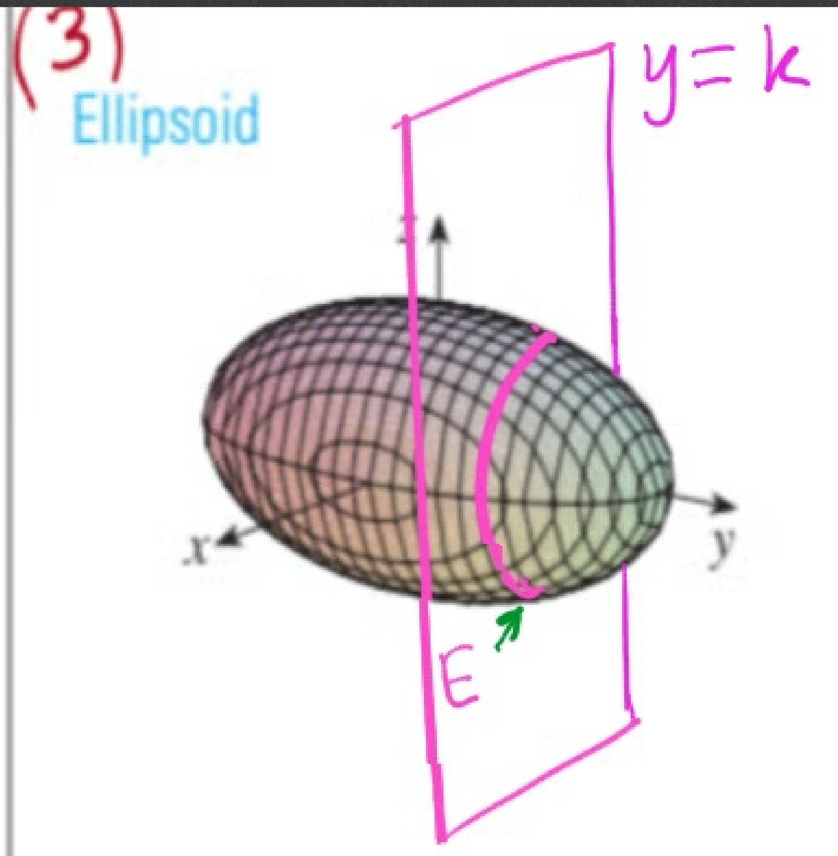
$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ ellipse in $\mathbb{R}^2$ $\Big\{$ $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ hyperbola in $\mathbb{R}^2$

TABLE 1 Graphs of quadric surfaces

trace = cross-section

Surface	Equation	Surface	Equation
(3) Ellipsoid 	$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ All traces are ellipses. If $a=b=c$, the ellipsoid is a sphere.	(4) Cone 	$\frac{z^2}{c^2}=\frac{x^2}{a^2}+\frac{y^2}{b^2}$ Horizontal traces are ellipses. Vertical traces in the planes $x=k$ and $y=k$ are hyperbolas if $k\neq 0$ but are pairs of lines if $k=0$.
(5a) Elliptic Paraboloid 	$\frac{z}{c}=\frac{x^2}{a^2}+\frac{y^2}{b^2}$ Horizontal traces are ellipses. Vertical traces are parabolas. The variable raised to the first power indicates the axis of the paraboloid.	(6a) Hyperboloid of One Sheet 	one minus sign $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ Horizontal traces are ellipses. Vertical traces are hyperbolas. The axis of symmetry corresponds to the variable whose coefficient is negative.
(5b) Hyperbolic Paraboloid 	$\frac{z}{c}=\frac{x^2}{a^2}-\frac{y^2}{b^2}$ Horizontal traces are hyperbolas. Vertical traces are parabolas. The case where $c<0$ is illustrated.	(6b) Hyperboloid of Two Sheets 	two minus signs $-\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ Horizontal traces in $z=k$ are ellipses if $k>c$ or $k<-c$. Vertical traces are hyperbolas. The two minus signs indicate two sheets.

(3) Ellipsoid



$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$

All traces are ellipses.

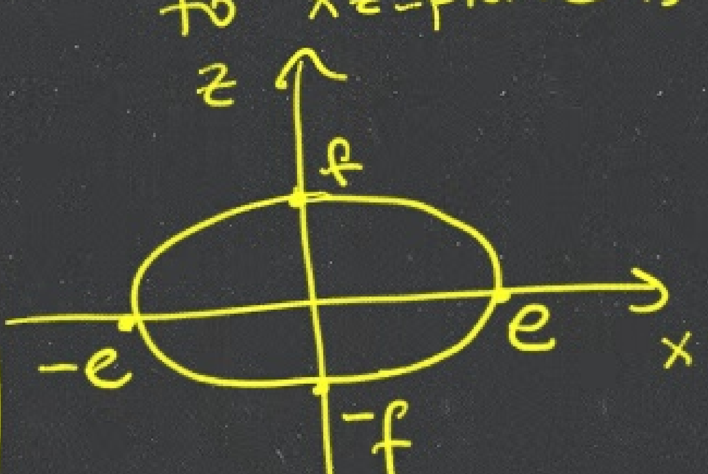
If $a=b=c$, the ellipsoid is a sphere.

$y=k \Rightarrow \frac{x^2}{a^2}+\frac{z^2}{b^2}=1-\frac{k^2}{c^2}=\frac{c^2-k^2}{c^2}$ for $c^2\geq k^2$

$\left(\frac{c^2}{c^2-k^2}\right)\left(\frac{x^2}{a^2}+\frac{z^2}{b^2}\right)=\left(\frac{c^2-k^2}{c^2}\right)\left(\frac{c^2}{c^2-k^2}\right)=1.$

$\frac{x^2}{\underbrace{a^2\cdot(c^2-k^2)/c^2}_{e^2}}+\frac{z^2}{\underbrace{b^2(c^2-k^2)/c^2}_{=f^2}}=1$

$\left\{\begin{array}{l}\frac{x^2}{e^2}+\frac{z^2}{f^2}=1\\ y=k\end{array}\right\}$ describes the pink ellipse E whose projection to xz-plane is



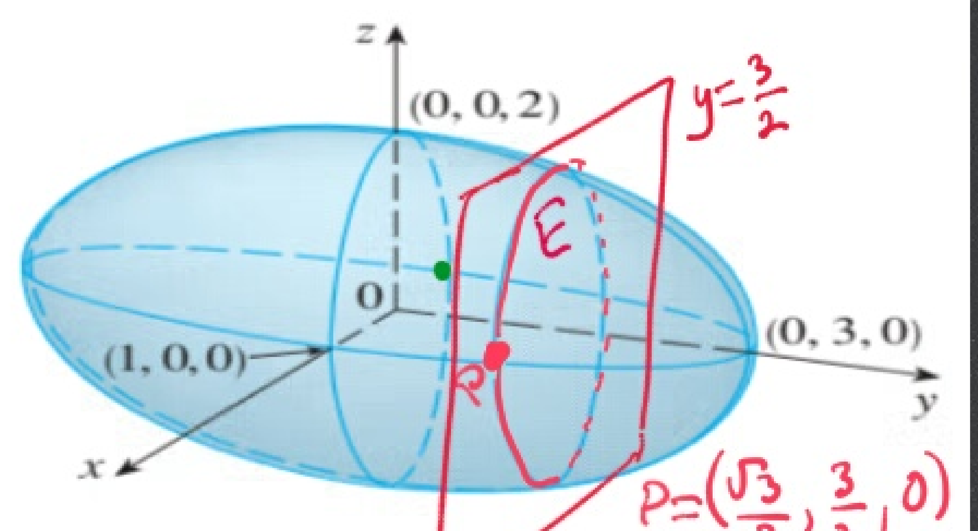


FIGURE 4

The ellipsoid $x^2+\frac{y^2}{9}+\frac{z^2}{4}=1$

$y=\frac{3}{2} \Rightarrow x^2+\frac{3^2}{2^2 3^2}+\frac{z^2}{4}=1 \Rightarrow x^2+\frac{z^2}{4}=\frac{3}{4}$

$\Rightarrow \frac{x^2}{3/4}+\frac{z^2}{3}=1$

$\Rightarrow \frac{x^2}{(\sqrt{3}/2)^2}+\frac{z^2}{(\sqrt{3})^2}=1$ is the projection of E on the xz-plane

$E:\left\{\begin{array}{l}y=\frac{3}{2}\\ \frac{x^2}{(\sqrt{3}/2)^2}+\frac{y^2}{(\sqrt{3})^2}=1\end{array}\right.$

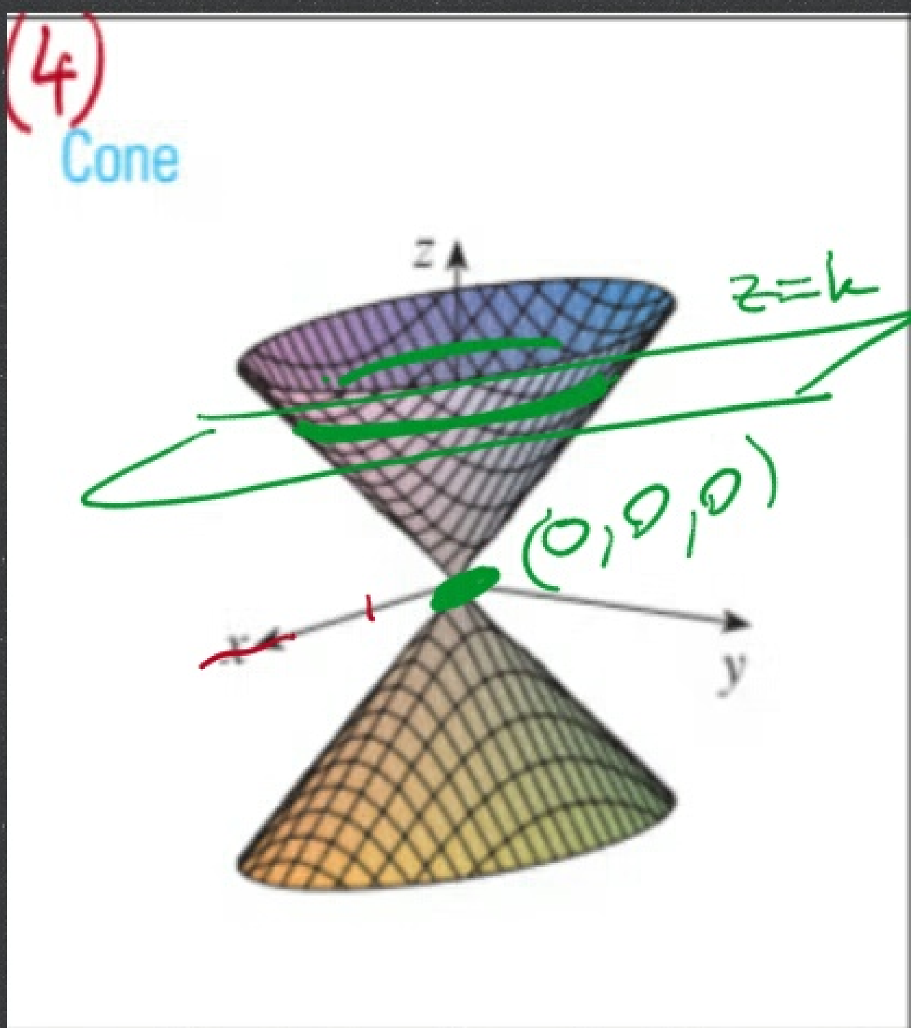
CONES:

$$S: \frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0 \iff \frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{z^2}{c^2}$$

Note that when $z=0$; $\frac{x^2}{a^2}+\frac{y^2}{b^2}=0$ is satisfied by $x=0=y$

so, S intersects with the xy-plane is $(0,0,0)$ only.

(4)



$$\frac{z^2}{c^2}=\frac{x^2}{a^2}+\frac{y^2}{b^2}$$

Horizontal traces are ellipses.

Vertical traces in the planes $x=k$ and $y=k$ are hyperbolas if $k\neq 0$ but are pairs of lines if $k=0$.

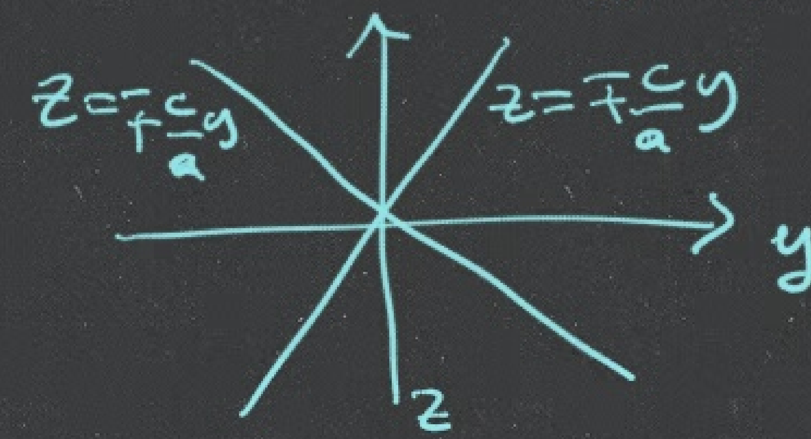
when $z=\mp k$, $\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{k^2}{c^2}$

$$\Updownarrow$$

$$\frac{x^2}{(ka/c)^2}+\frac{y^2}{(kb/c)^2}=1$$ ellipse for $k\neq 0$.

so, all horizontal traces are ellipses.

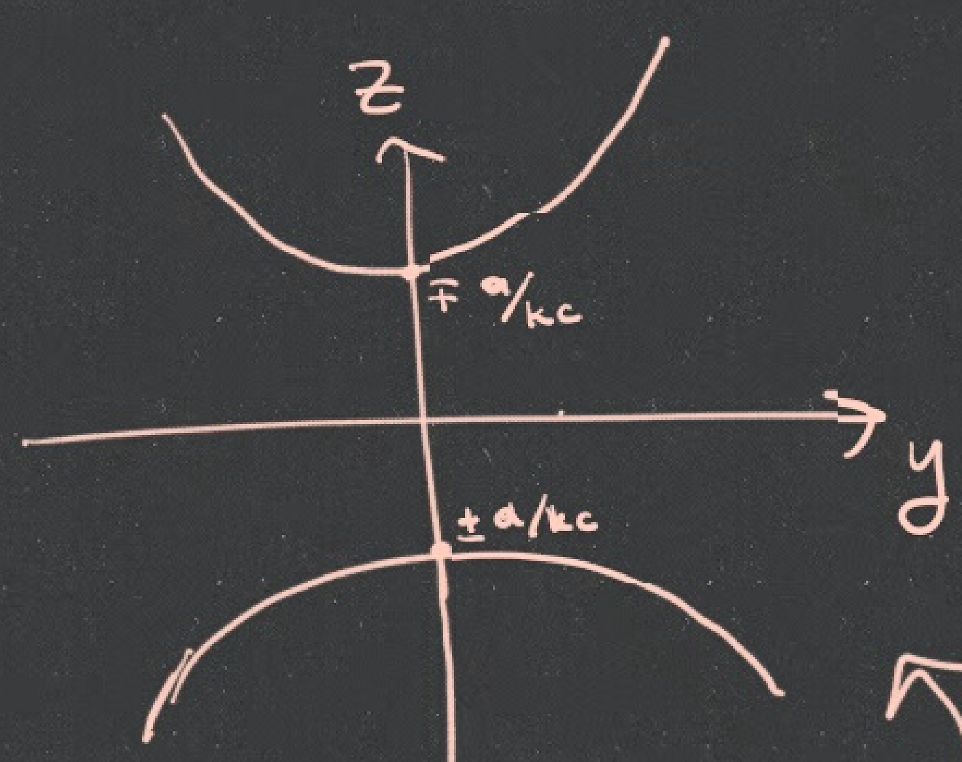
when $x=0$; $\frac{z^2}{c^2}=\frac{y^2}{a^2} \Rightarrow \frac{z^2}{c^2}-\frac{y^2}{a^2}=\underbrace{\left(\frac{z}{c}-\frac{y}{a}\right)}_{=0}\underbrace{\left(\frac{z}{c}+\frac{y}{a}\right)}_{=0}=0$, so $\boxed{z=\frac{c}{a}y}$ $\boxed{z=-\frac{c}{a}y}$ are lines in the yz-plane



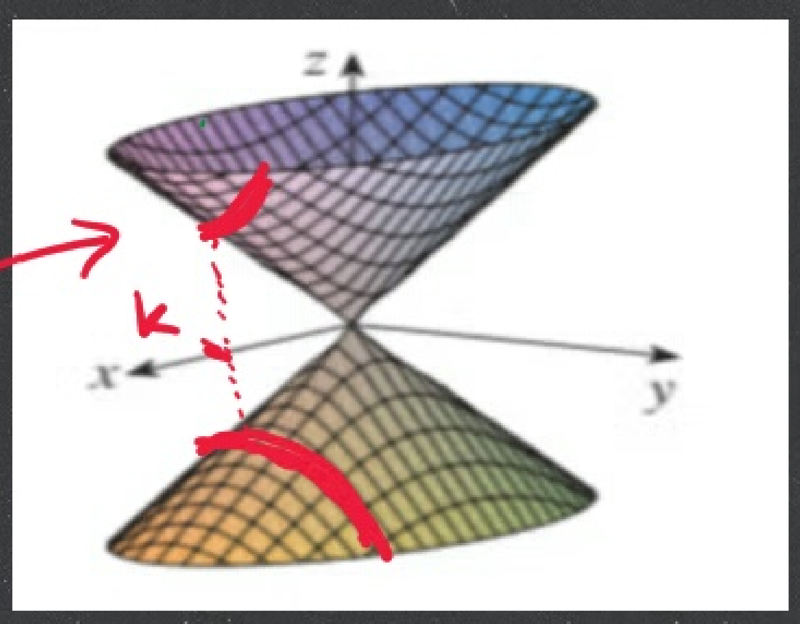
when $x=k$; $\frac{z^2}{c^2}=\frac{k^2}{a^2}+\frac{y^2}{b^2}$

hyperbola $\frac{z^2}{c^2}-\frac{y^2}{b^2}=\frac{k^2}{a^2}$

$$\frac{z^2}{(kc/a)^2}-\frac{y^2}{(kb/a)^2}=1$$

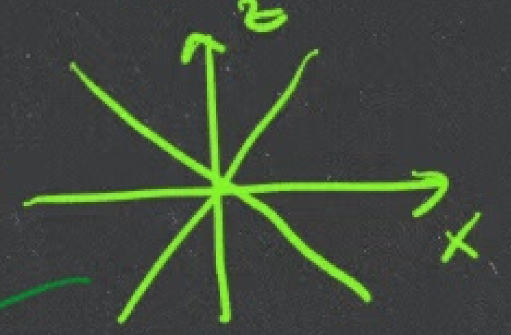


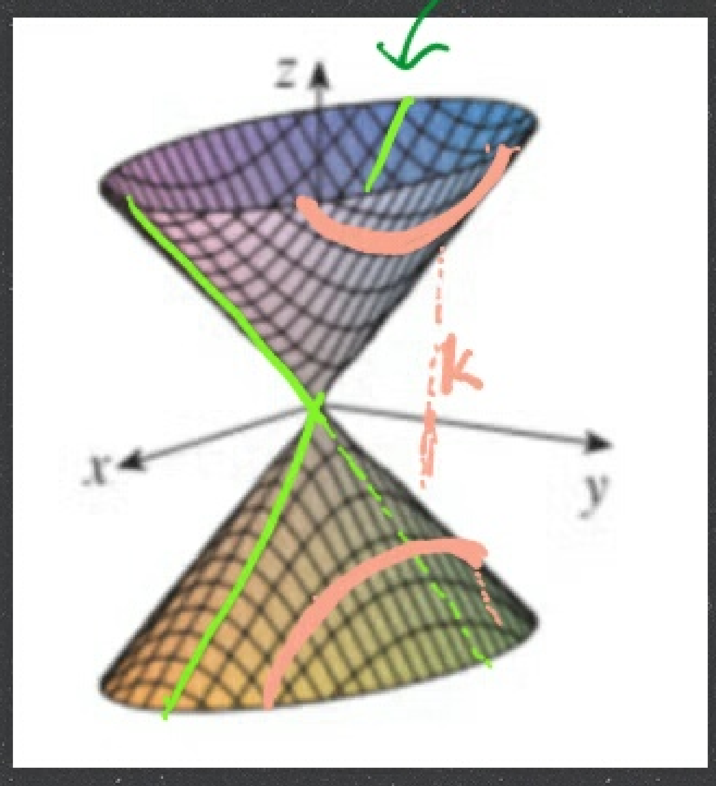
whose projection to yz-plane is

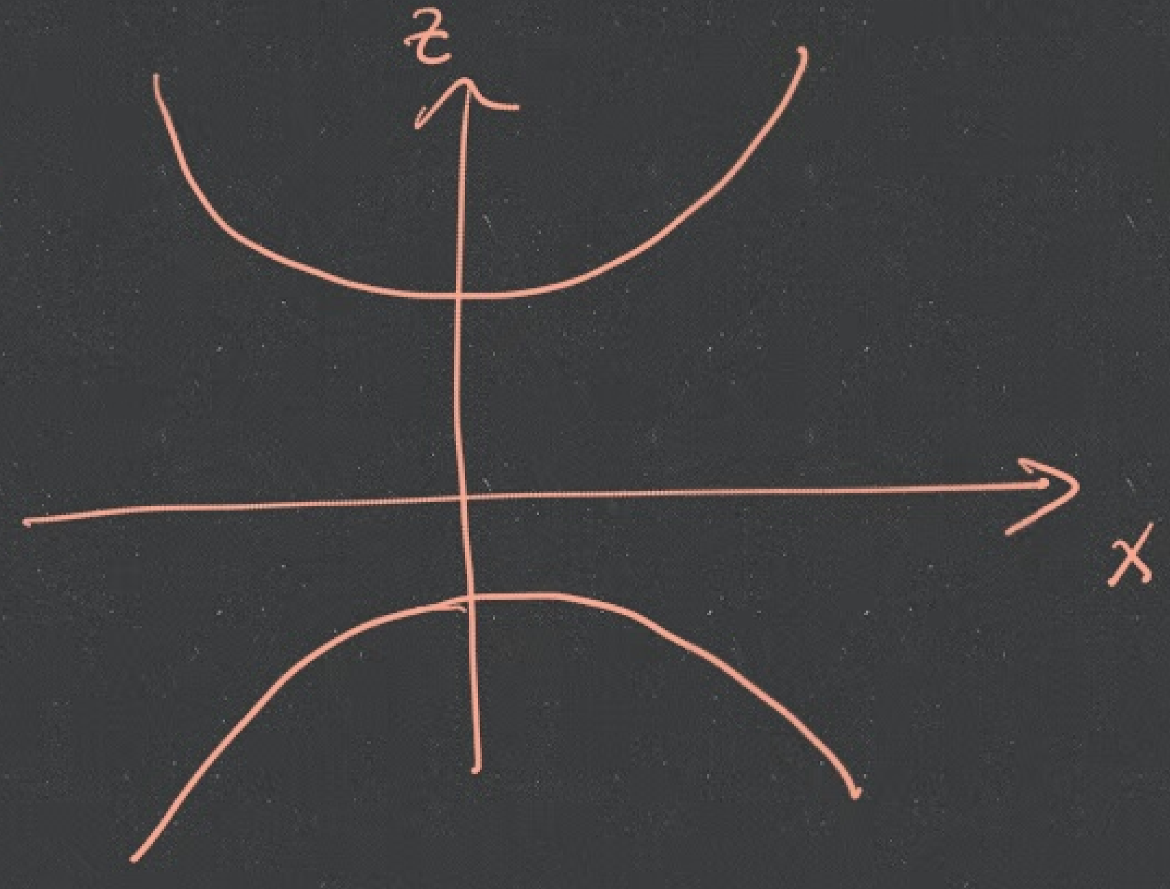


when $y=k$, $\frac{z^2}{c^2}=\frac{x^2}{a^2}+\frac{k^2}{b^2} \iff \frac{z^2}{c^2}-\frac{x^2}{a^2}=\frac{k^2}{b^2} \xRightarrow{k\neq 0} \frac{z^2}{(kc/b)^2}-\frac{x^2}{(ka/b)^2}=1$

$k=0$, $\frac{z^2}{c^2}=\frac{x^2}{a^2}$,







hyperbola again! whose projection to xz-plane vertical traces are parabolas $x=k,\ y=k$

5) Paraboloids

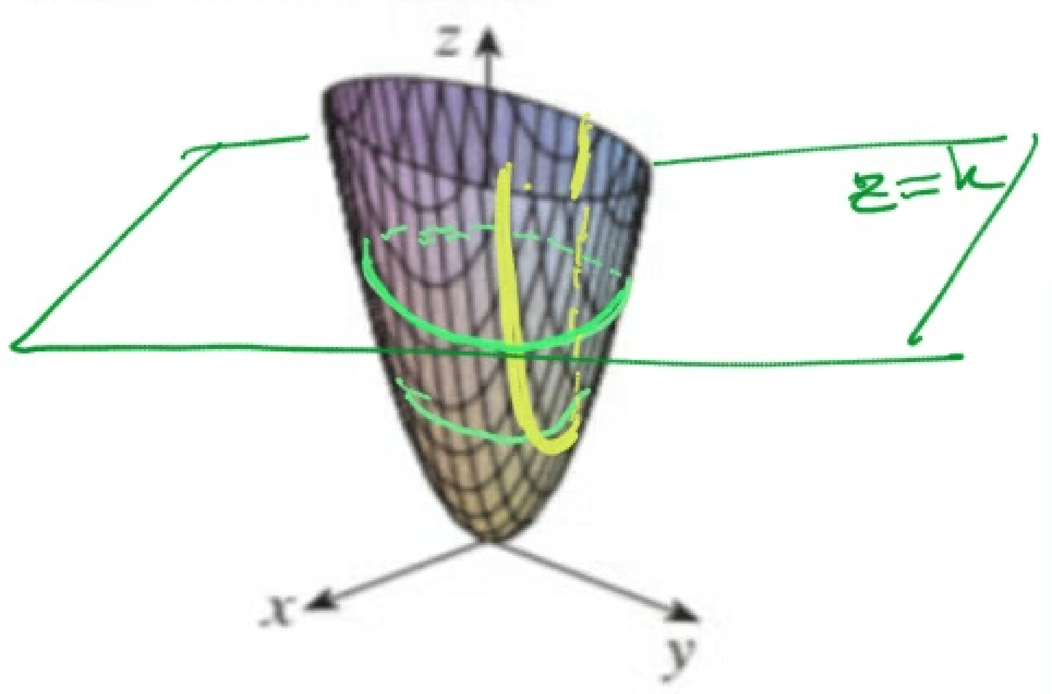
elliptic (intersection with horizontal planes is ellipses) horizontal traces $z=k$

$\frac{z}{c} = \frac{x^2}{a^2} \pm \frac{y^2}{b^2}$

hyperbolic (intersection with horizontal planes are hyperbolas) horizontal traces $z=k$

5a

Elliptic Paraboloid



$$\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

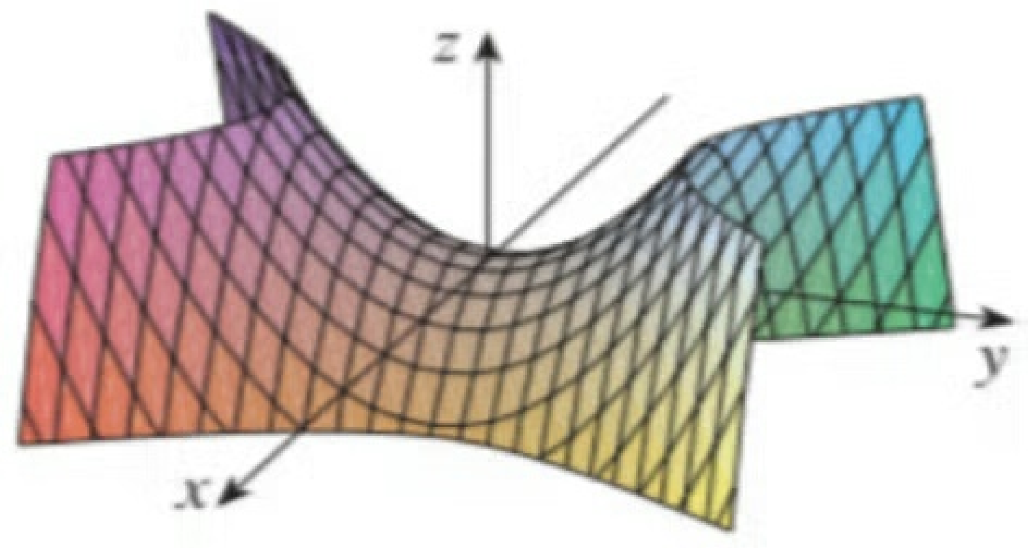
Horizontal traces are ellipses.

Vertical traces are parabolas.

The variable raised to the first power indicates the axis of the paraboloid.

5b

Hyperbolic Paraboloid



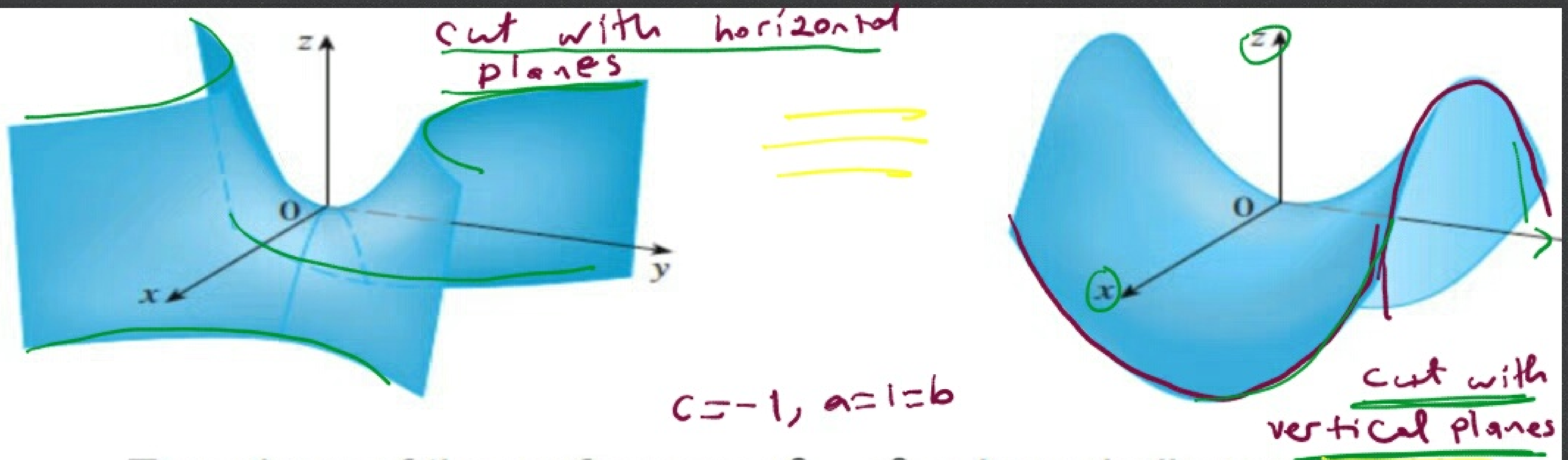
$$\frac{z}{c} = \frac{x^2}{a^2} - \frac{y^2}{b^2}$$

Horizontal traces are hyperbolas.

Vertical traces are parabolas.

The case where $c < 0$ is illustrated.

saddle



saddle

Two views of the surface $z = y^2 - x^2$, a hyperbolic paraboloid.

EXAMPLE 5 Sketch the surface $z = y^2 - x^2$.

SOLUTION The traces in the vertical planes $x = k$ are the parabolas $z = y^2 - k^2$, which open upward. The traces in $y = k$ are the parabolas $z = -x^2 + k^2$, which open downward. The horizontal traces are $y^2 - x^2 = k$, a family of hyperbolas. We draw the families of traces in Figure 6, and we show how the traces appear when placed in their correct planes in Figure 7.

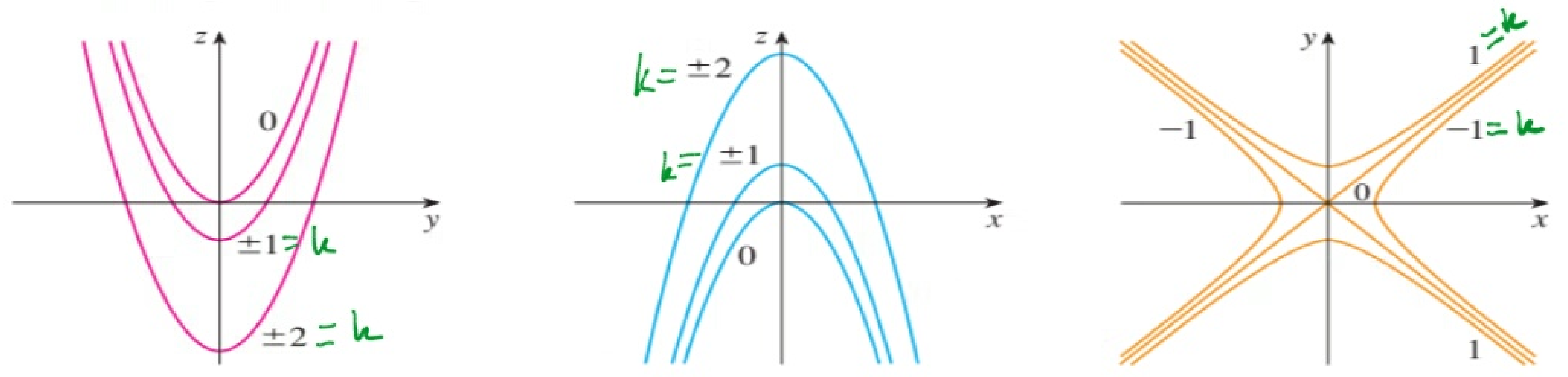


FIGURE 6
Vertical traces are parabolas; horizontal traces are hyperbolas. All traces are labeled with the value of k.

Traces in $x = k$ are $z = y^2 - k^2$ Traces in $y = k$ are $z = -x^2 + k^2$ Traces in $z = k$ are $y^2 - x^2 = k$

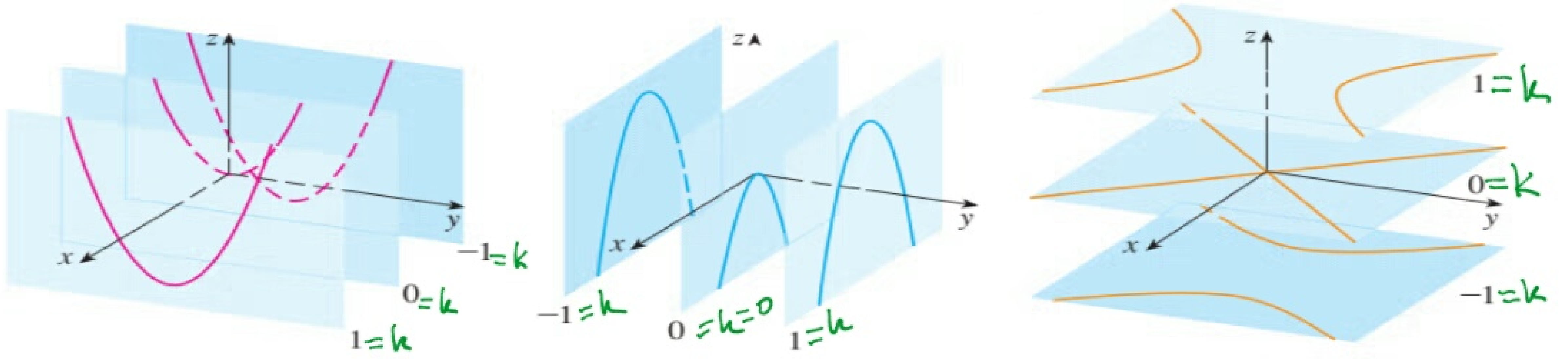


FIGURE 7
Traces moved to their correct planes

Traces in $x = k$ Traces in $y = k$ Traces in $z = k$

Vertical traces are hyperbolas, horizontal traces are ellipses.

6) Hyperboloids — one sheet $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ when $x=0$

— two sheets $-\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$

one sheeted

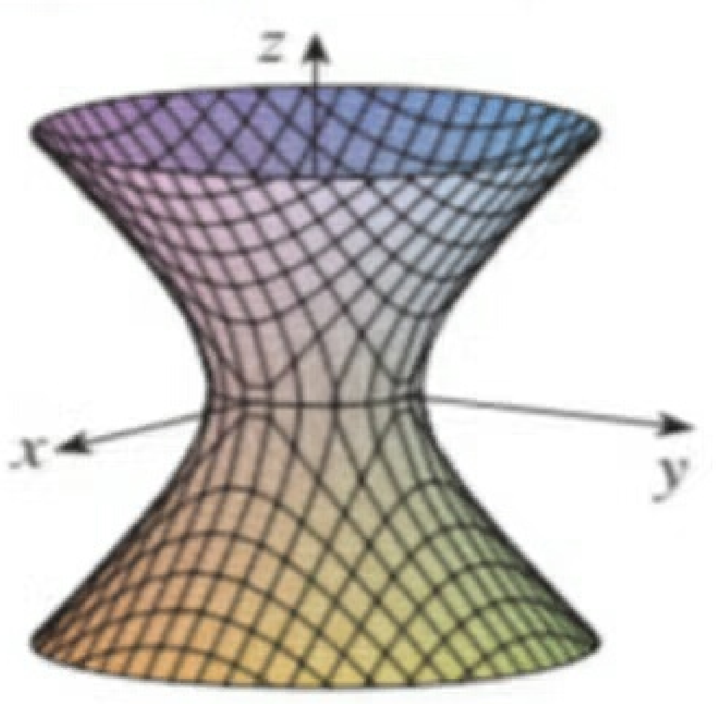
$z=k$; $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1+\frac{k^2}{c^2}=\frac{c^2+k^2}{c^2}$ ellipse;

two sheeted

$-\frac{x^2}{a^2}-\frac{y^2}{b^2}=1-\frac{k^2}{c^2}=\frac{c^2-k^2}{c^2} \Leftrightarrow \underbrace{\frac{x^2}{a^2}+\frac{y^2}{b^2}}_{\text{ellipse}}=\frac{k^2-c^2}{c^2} \geq 0$ has no sol. when $|c|>|k|$

6a

Hyperboloid of One Sheet



$x=0=y$ gives no solution.

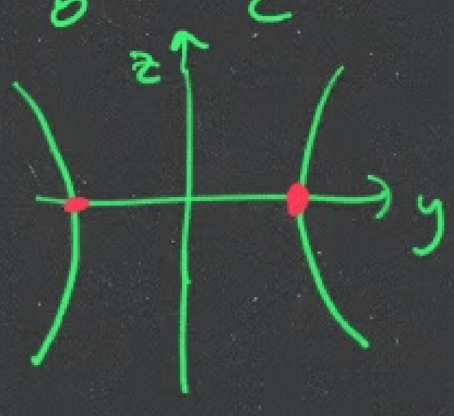
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$$

Horizontal traces are ellipses.

Vertical traces are hyperbolas.

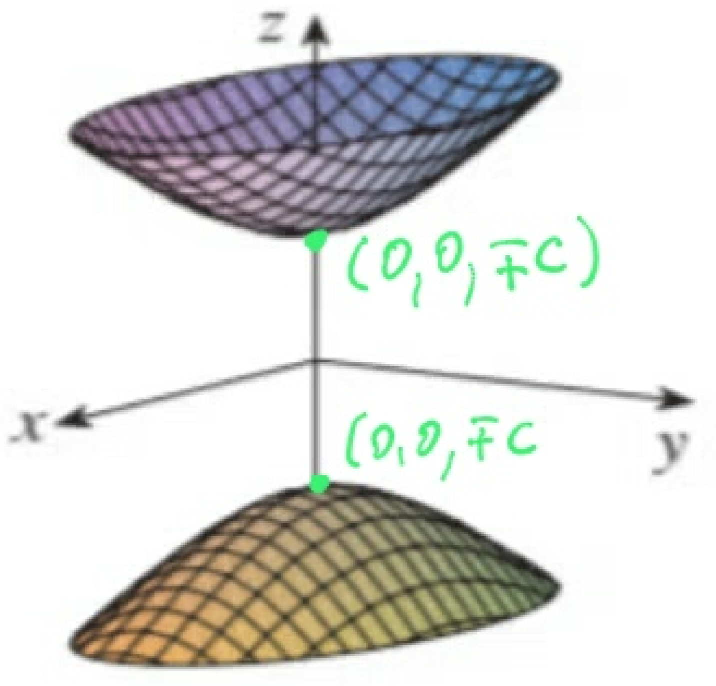
when $x=0$ (vertical trace)

$\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ hyperbola



6b

Hyperboloid of Two Sheets



$x=0=y$ gives two solutions $z=\mp c$

$$-\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

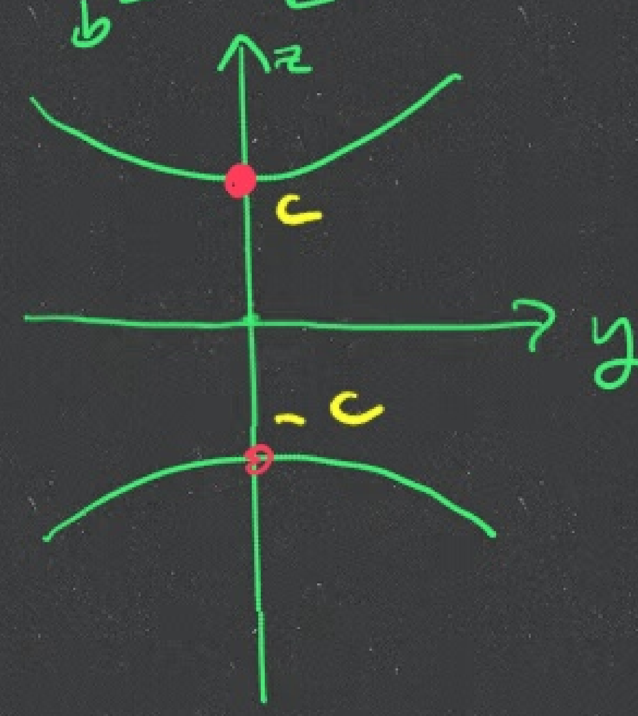
Horizontal traces in $z=k$ are ellipses if $k>c$ or $k<-c$.

Vertical traces are hyperbolas.

The two minus signs indicate two sheets.

when $x=0$, (vertical trace)

$-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ hyperbola

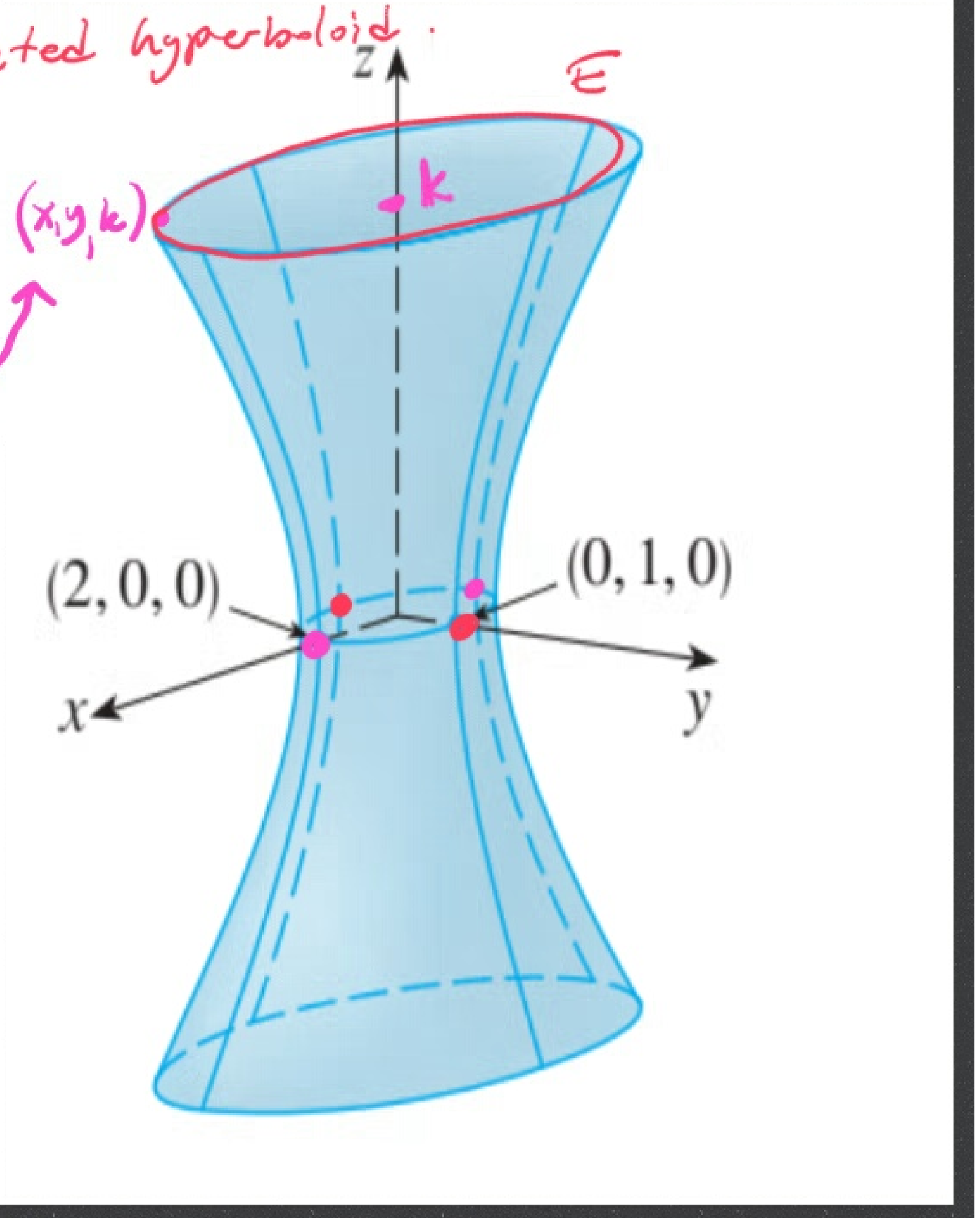


EXAMPLE 6 Sketch the surface $\frac{x^2}{4}+y^2-\frac{z^2}{4}=1$.

one sheeted hyperboloid.

SOLUTION The trace in any horizontal plane $z=k$ is the ellipse

$$\frac{x^2}{4}+y^2=1+\frac{k^2}{4}, \quad z=k$$



$x=0=y \Rightarrow$ no solution

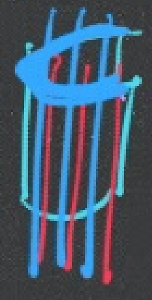
$x=0=z \Rightarrow y=\mp 1 \Rightarrow (0,1,0),(0,-1,0)$ are on it

$y=0=z \Rightarrow x=\mp 2 \Rightarrow (2,0,0),(-2,0,0)$ are on it

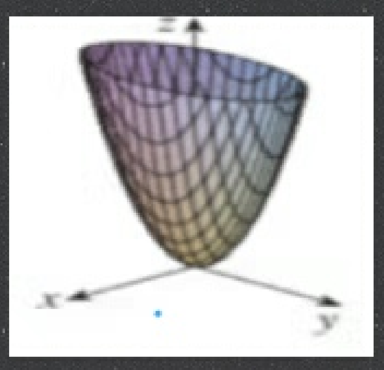
Which quadric surfaces do contain a line?

Let's list the quadric surfaces in a different order and see which contain lines.

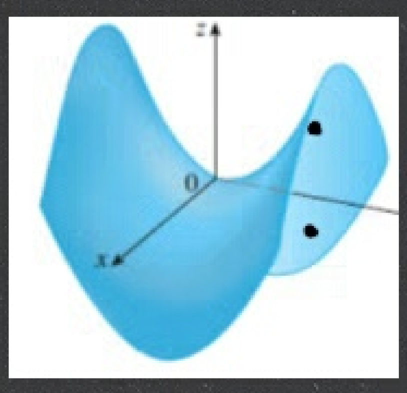
- cylinders have only two variables, they contain lines parallel to the axis of the missing variable.



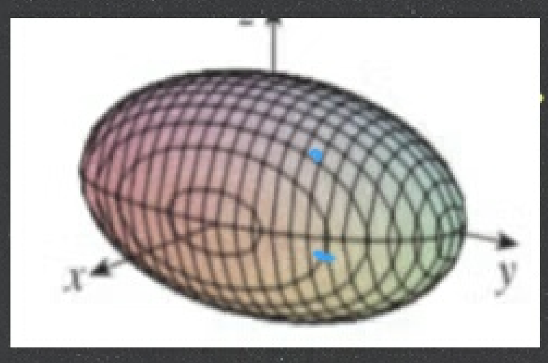
- paraboloids
  - elliptic $f(x,y) = z = x^2 + y^2$



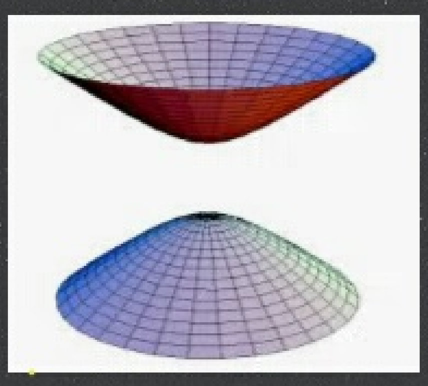
  - hyperbolic $f(x,y) = z = x^2 - y^2$
    saddle



- ellipsoid (sphere like) $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

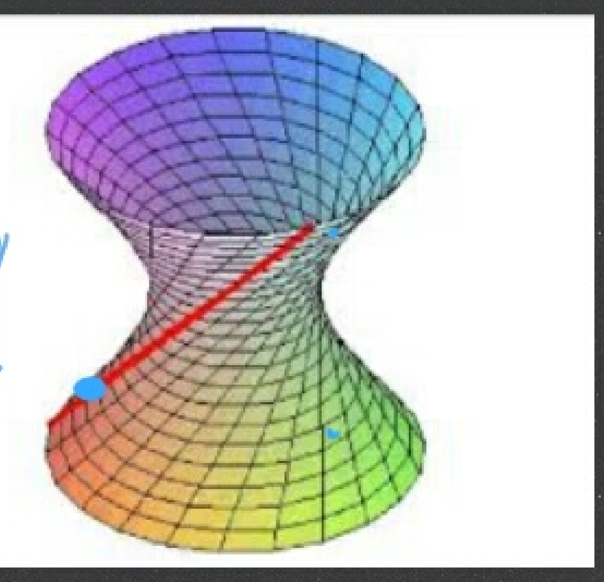


- hyperboloid
  - two sheeted $-x^2 - y^2 + z^2 = 1$



  - $x^2 + y^2 - z^2 = 1$ one sheeted

$\mathcal{H}$



$\mathcal{H}$ contains lines $(*)$

Proof of $(*)$

Let r be a real number, and $P_r = (r, 0, \sqrt{r^2-1})$ be a point in $\mathbb{R}^3$.
P_r satisfies the equation $x^2+y^2-z^2=1$ because $r^2 + 0^2 - (\sqrt{r^2-1})^2 = \cancel{r^2} - \cancel{r^2} + 1 = \underline{1}$
then P_r is on the hyperboloid $\mathcal{H}$.

Let $\vec{v}_r = \sqrt{r^2-1}\,\vec{i} + \vec{j} + r\,\vec{k}$, then $\vec{OP_r} + t\vec{v}_r$ is a line $\mathcal{L}_r$.

$\Rightarrow \mathcal{L}_r(t) = (\underbrace{r + t\sqrt{r^2-1}}_{x}, \underbrace{t}_{y}, \underbrace{\sqrt{r^2-1} + tr}_{z})$. Let's check if $\mathcal{L}_r(t) \in \mathcal{H}$.

$x^2+y^2-z^2 = (r + t\sqrt{r^2-1})^2 + t^2 - (\sqrt{r^2-1} + tr)^2 =$
$r^2 + \cancel{2tr\sqrt{r^2-1}} + t^2(\cancel{r^2}-1) + t^2 - (\cancel{r^2}-1 + \cancel{2tr\sqrt{r^2-1}} + \cancel{t^2r^2}) = \underline{1}$

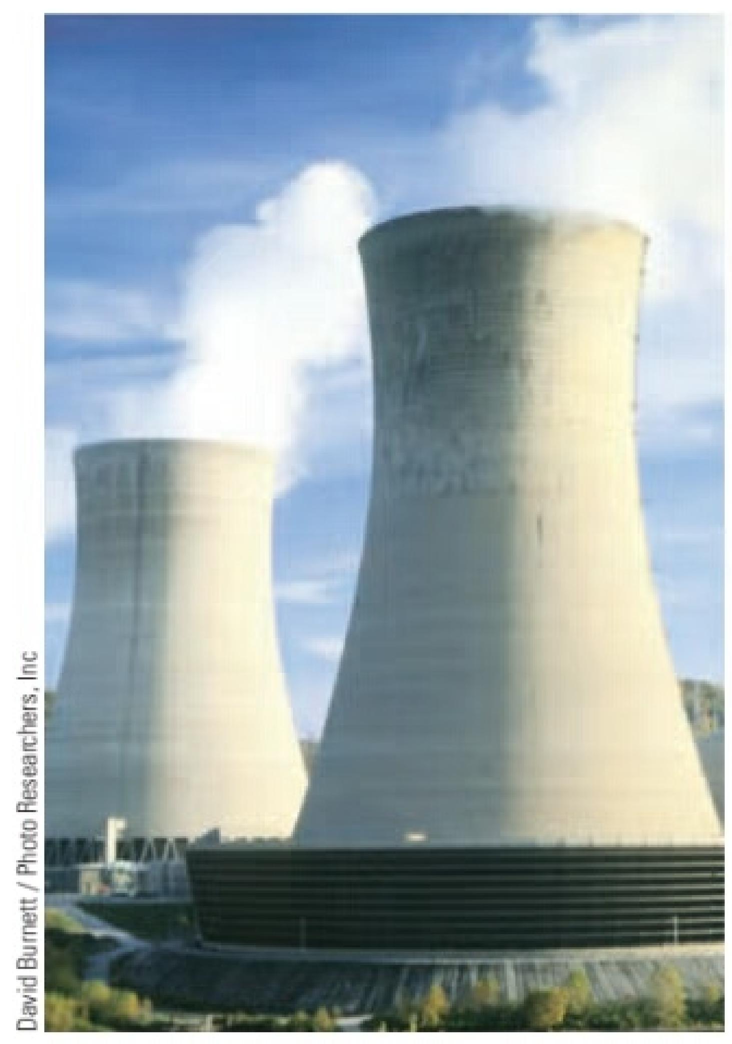
$\Rightarrow$ every point on $\mathcal{L}_r$ is on $\mathcal{H}$ for any $r \in \mathbb{R}$.
so $\mathcal{H}$ contains infinitely many lines

Hyperbolid gear transmission →

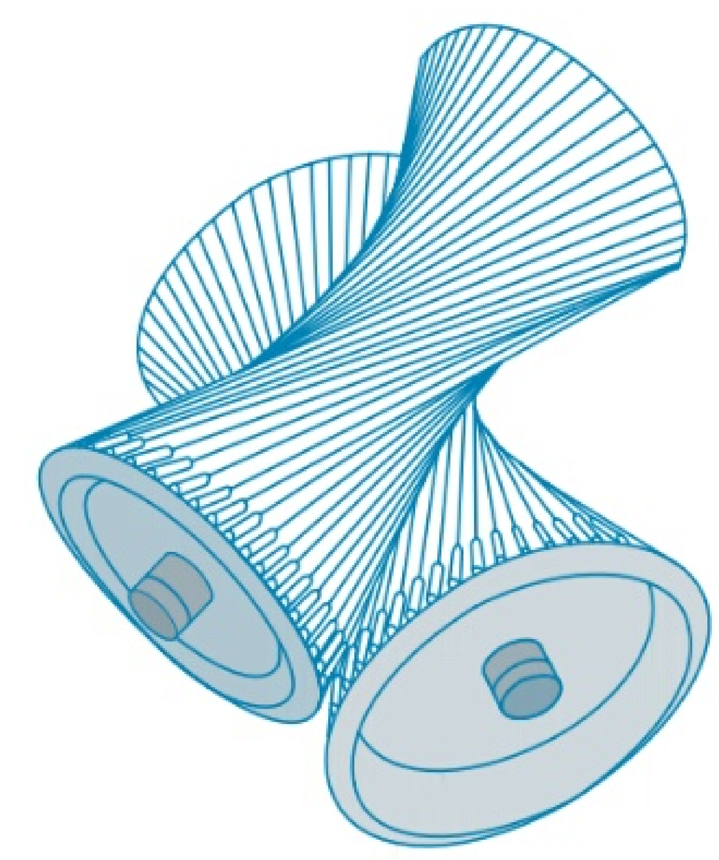
There are animations on the YouTube showing this type of gears transmitting the motion from one direction to another.

transmits the rotation by $\pi/4$

A satellite dish reflects signals to the focus of a paraboloid.



Nuclear reactors have cooling towers in the shape of hyperboloids.

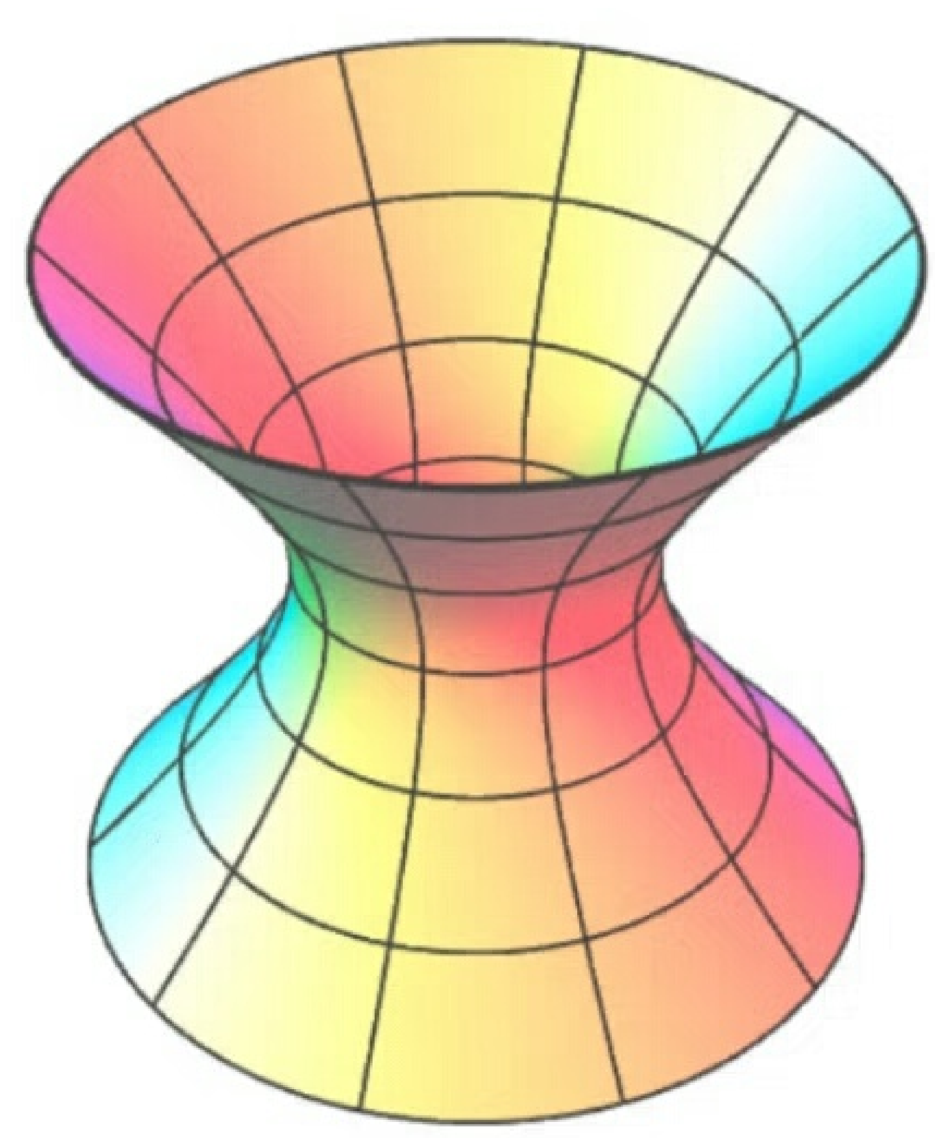


Hyperboloids produce gear transmission.

Interactive Gallery Of Quadric Surfaces

Intro | Elliptic Paraboloid | Hyperbolic Paraboloid | Ellipsoid | Double Cone | Hyperboloid of One Sheet | Hyperboloid of Two Sheets

Quadric surfaces are important objects in Multivariable Calculus and Vector Analysis classes. We like them because they are natural 3D-extensions of the so-called *conics* (ellipses, parabolas, and hyperbolas), and they provide fairly nice surfaces to use as examples for the rest of your class.



The basic quadric surfaces are described by the following five equations, where A, B, and C are constants.

$$z = Ax^2 + By^2 \qquad z^2 = Ax^2 + By^2 \qquad \frac{x^2}{A^2} + \frac{y^2}{B^2} + \frac{z^2}{C^2} = 1$$

https://faculty.math.illinois.edu/~nmd/quadrics/