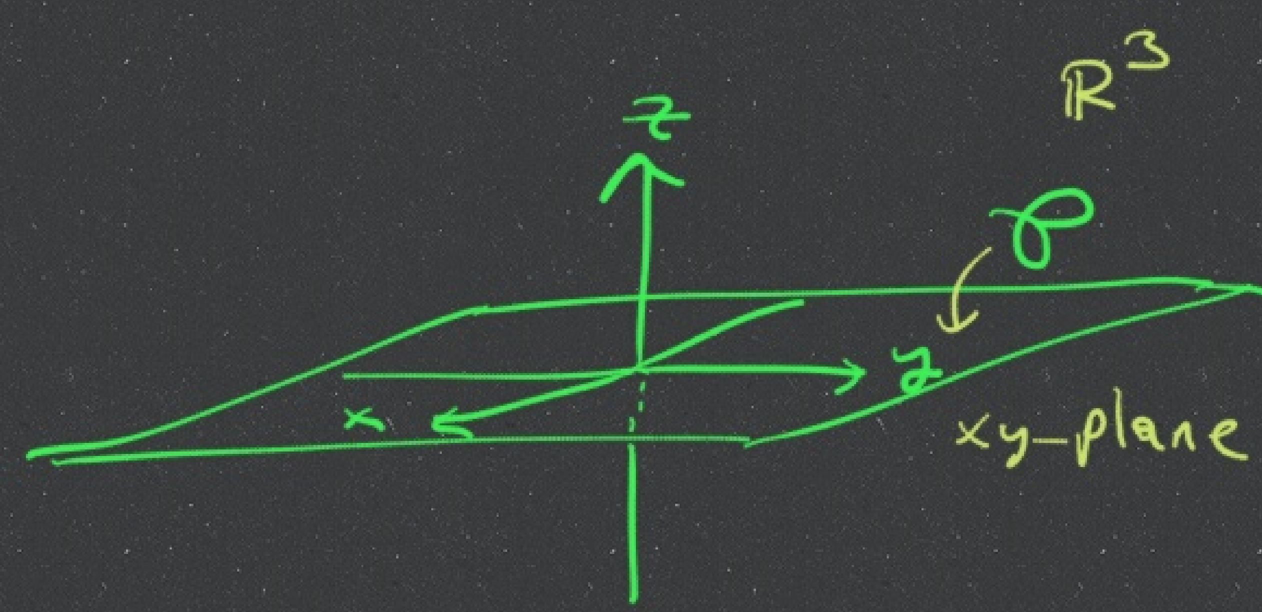
10.4. PLANES & LINES

Planes in 3-space; some examples:

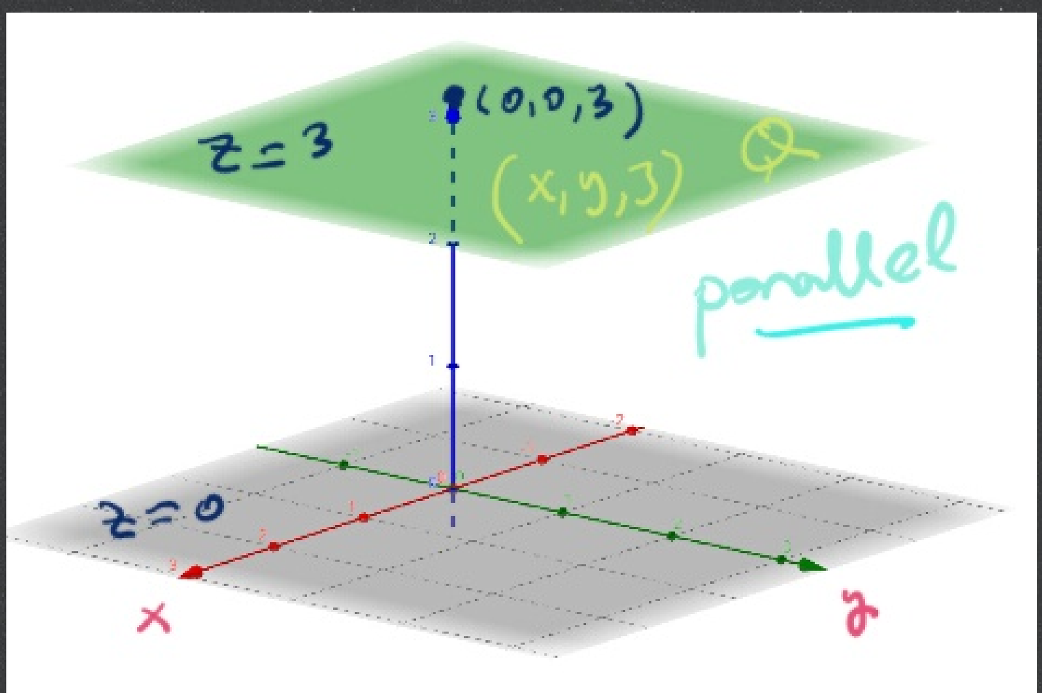


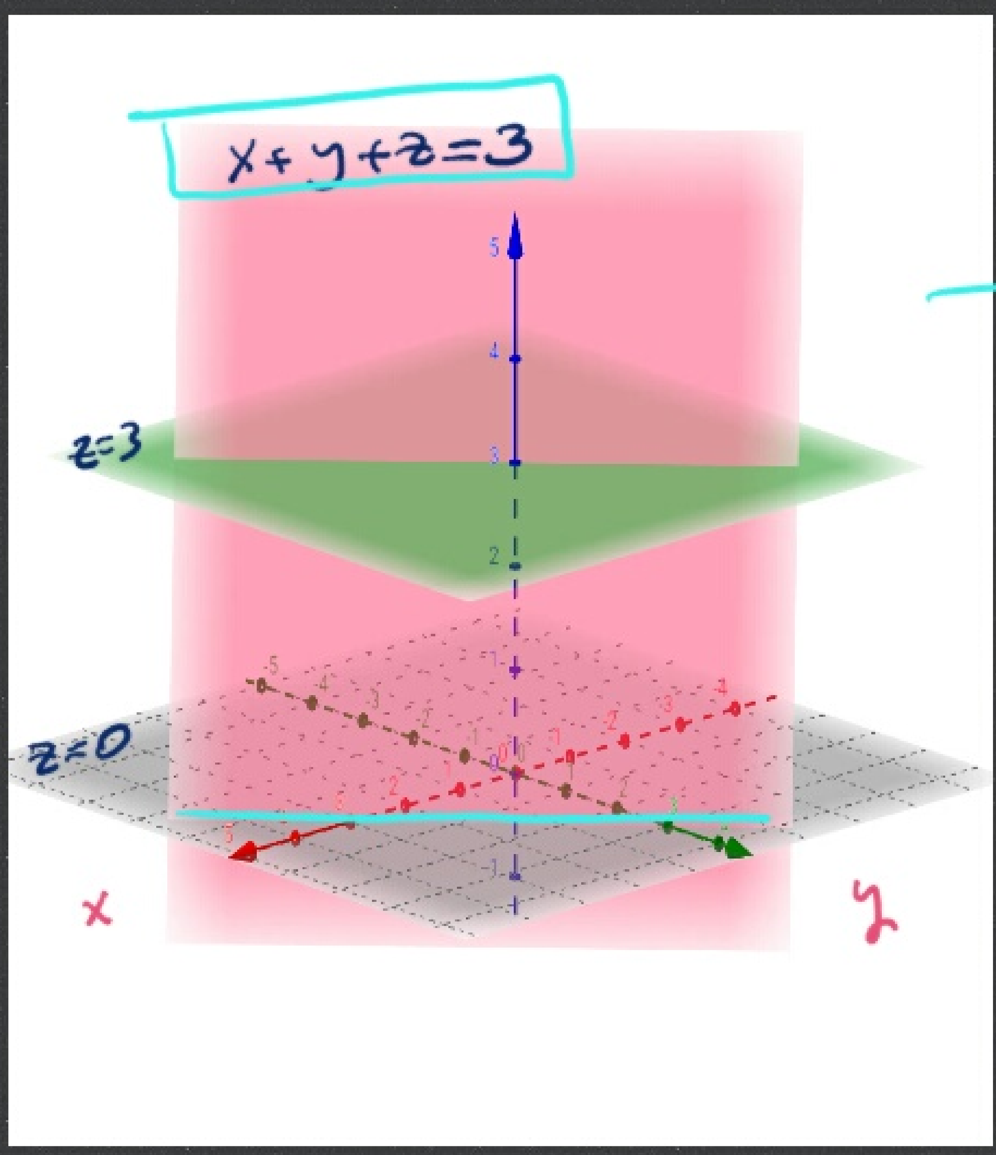
1) $z=0$ describes a plane in $\mathbb{R}^3$

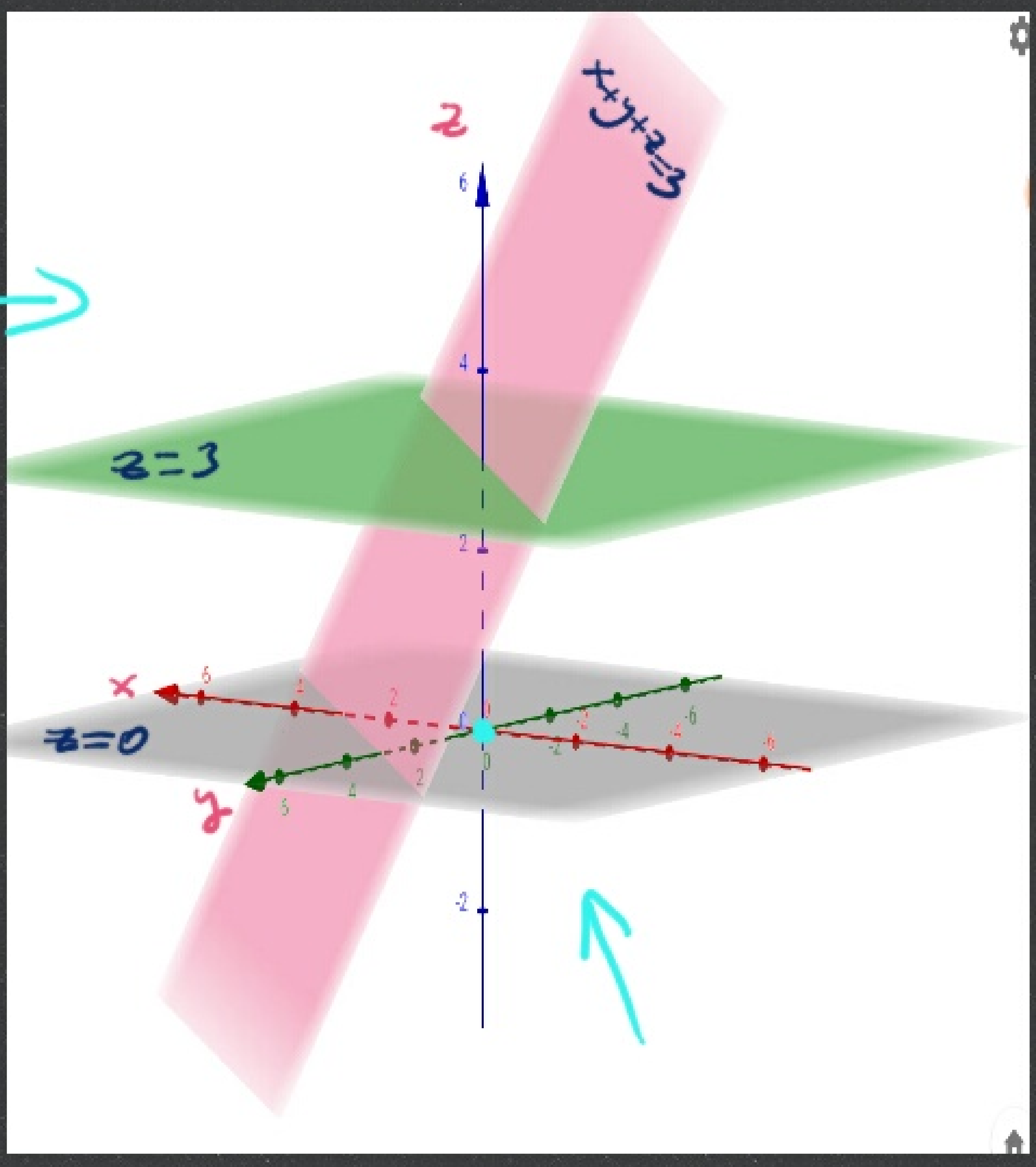
$\mathcal{P} = \{(x,y,z) \in \mathbb{R}^3 \mid z=0\} = \{(x,y,0) \mid x,y \in \mathbb{R}\}$

(2) $z=3$ describes a plane in $\mathbb{R}^3$

$\mathcal{Q} = \{(x,y,3) \mid x,y \in \mathbb{R}\}$





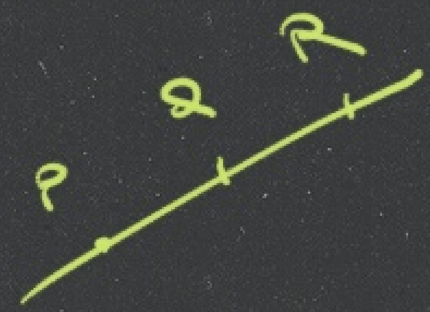


Remark:

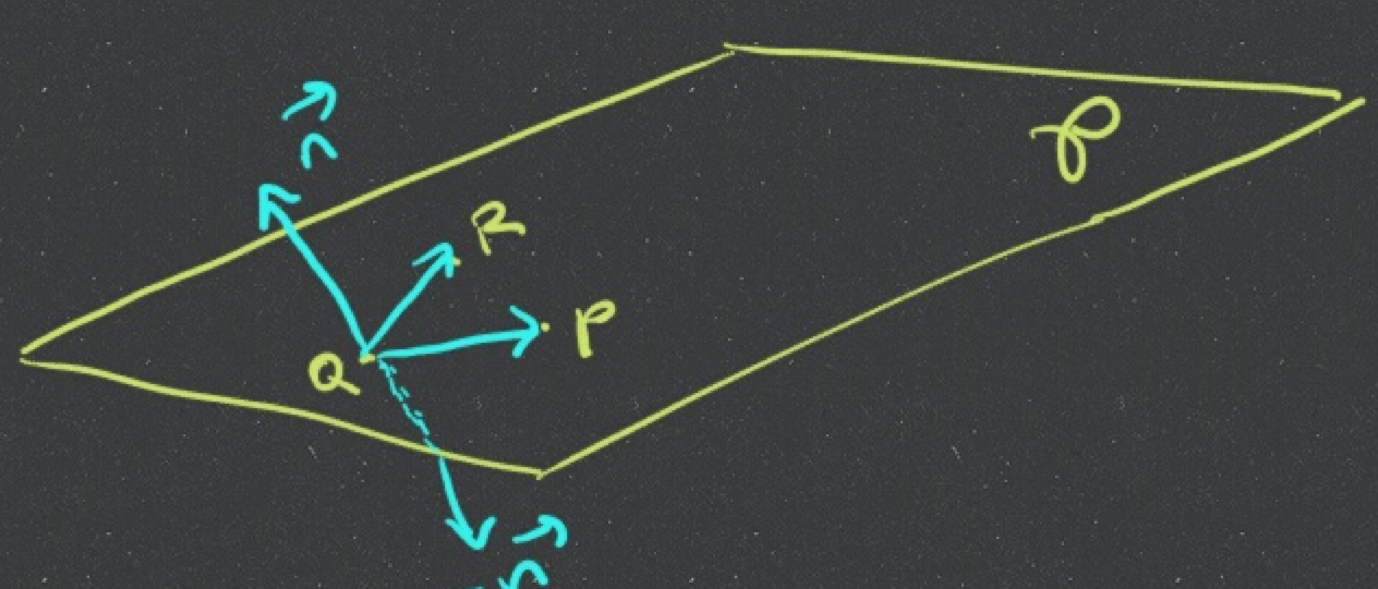
① There is a unique (exactly one) LINE passing through any two distinct points P, Q in $\mathbb{R}^3$.

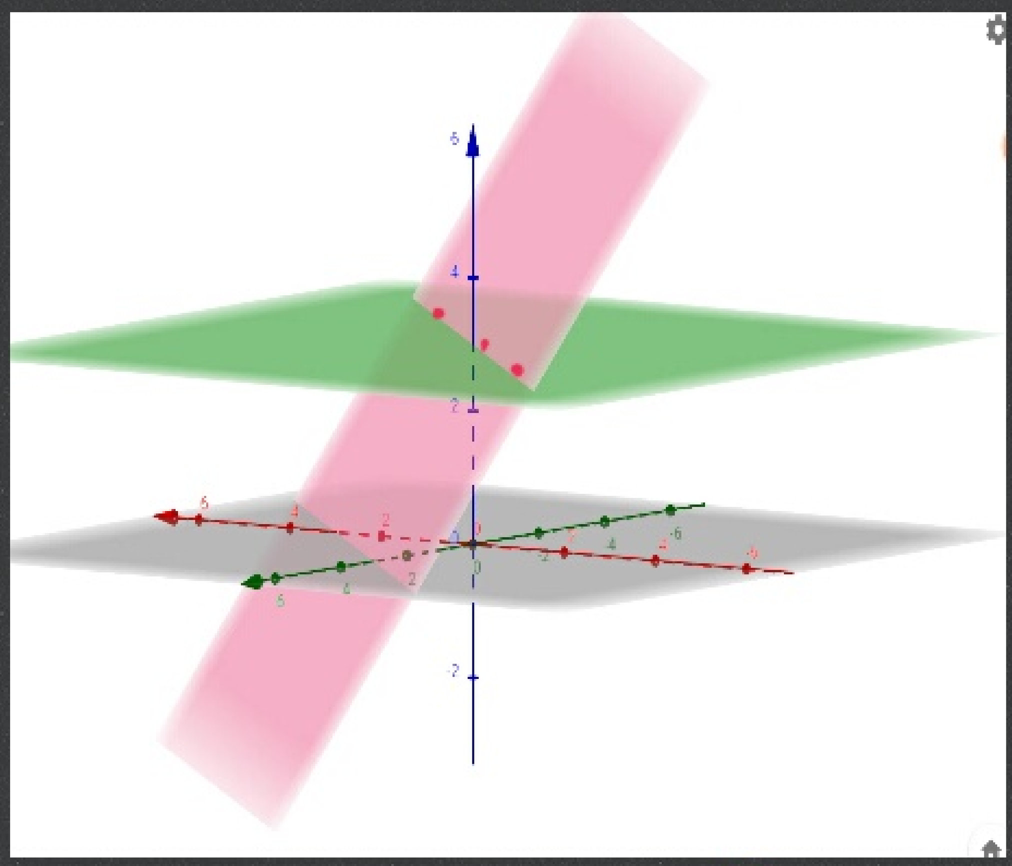
② There is a unique (exactly one) PLANE passing through any three points P, Q, R in $\mathbb{R}^3$ if they are not on the same line.

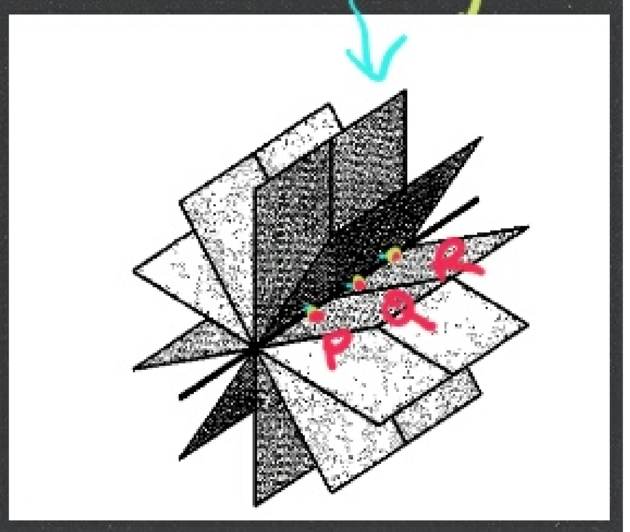
(If P, Q, R on the same line



then, there are infinitely many planes containing P, Q, R)



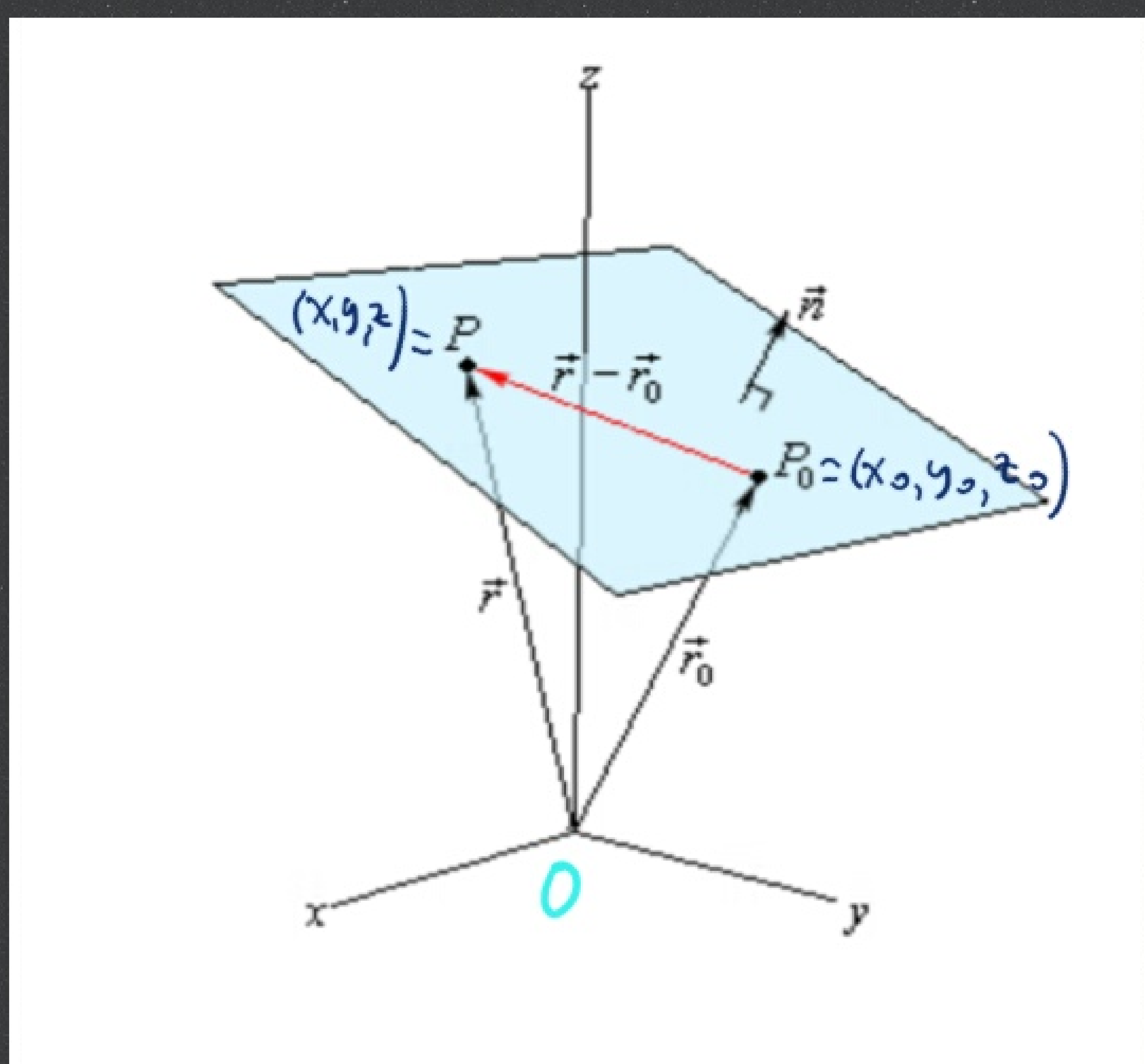




Moreover, there is a unique nonzero vector, $\vec{n}$, called the normal vector for the plane determined by P, Q R ; $\vec{n} = \overrightarrow{QP} \times \overrightarrow{QR} \neq \vec{0}$

③ A normal vector $\vec{n}$ and a point P_0 determines a plane. Moreover $-\vec{n}$ and P_0 determines the same plane.

What is the most general form of a plane equation in $\mathbb{R}^3$?.



suppose $\vec{n} = A\vec{i} + B\vec{j} + C\vec{k}$ and $P_0 = (x_0, y_0, z_0)$

Observation:

Let $P = (x, y, z)$ be an arbitrary point of the plane.

Let $\vec{r} = \overrightarrow{OP}$, $\vec{r}_0 = \overrightarrow{OP_0}$

Then $\vec{r} - \vec{r}_0 = (x - x_0)\vec{i} + (y - y_0)\vec{j} + (z - z_0)\vec{k}$.

Then $\vec{r} - \vec{r}_0 \perp \vec{n}$.

$\Updownarrow$

Plane equation. $\boxed{(\vec{r} - \vec{r}_0) \cdot \vec{n} = 0 = \vec{n} \cdot (\vec{r} - \vec{r}_0)}$

$\Leftrightarrow \quad A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$

$Ax + By + Cz - Ax_0 - By_0 + Cz_0 = 0$

$\Leftrightarrow \quad Ax + By + Cz - Ax_0 + By_0 + Cz_0 = D \in \mathbb{R}$

. Any equation of the form $Ax + By + Cz = D$ defines a plane whose normal vector $\vec{n} = A\vec{i} + B\vec{j} + C\vec{k}$.

Example ① $1 \cdot x + 1 \cdot y + 1z = 3$ above, where $\vec{n} = 1 \cdot \vec{i} + 1\vec{j} + 1\vec{k}$.
and passes through $(0,0,3)$ (or $(0,3,0)$, $(3,0,0)$, $(1,1,1)$, $(1,2,0)$ etc.....)

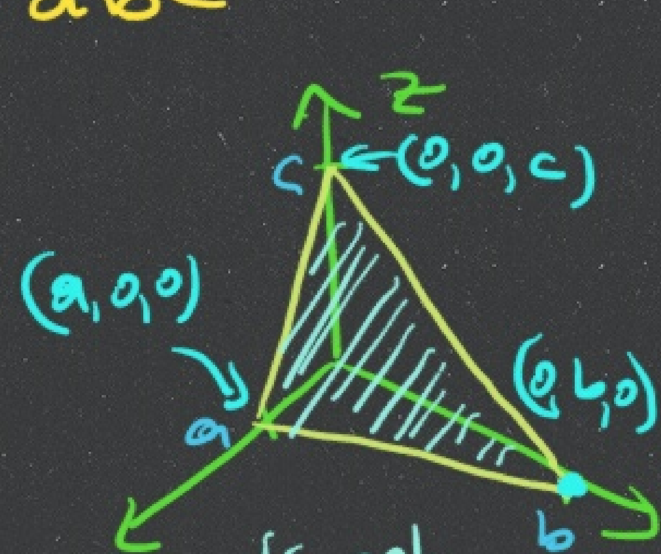
② $z = 3$ determines a plane whose normal vector is $\vec{n} = 0 \cdot \vec{i} + 0 \cdot \vec{j} + 3\vec{k} = 3\vec{k}$ and $(x, y, 3)$ is on the plane for any $x, y \in \mathbb{R}$.

④ $\dfrac{x}{a} + \dfrac{y}{b} + \dfrac{z}{c} = 1 \Leftrightarrow bcx + acy + abz = abc$

$x = 0 = y \Rightarrow z = c \Rightarrow$ z intercept is c

$x = 0 = z \Rightarrow y = b \Rightarrow$ y " " b

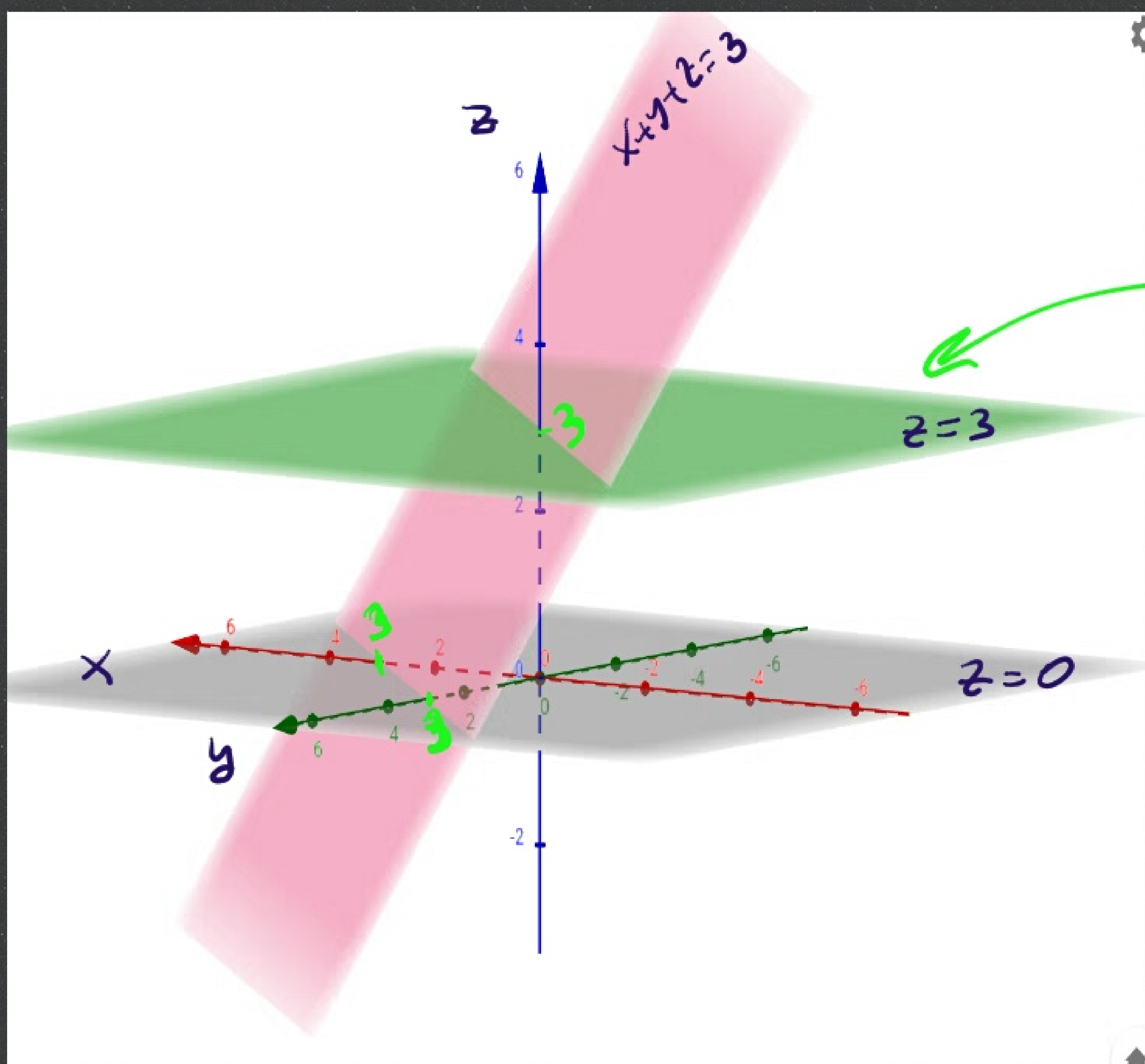
$y = 0 = z \Rightarrow x = a \Rightarrow$ x " " a



Is only a portion of the plane

intercept equation of a plane
(intercept form of the) equation.

Intercept points can be finitely many or infinitely many.



Let's reexamine the previous examples.

1) $z=3$, Green plane

has only z-intercept $z=3$,

no x-intercept,

no y-intercept.

2) $x+y+z=3$ pink plane

$x=y=0 \Rightarrow z=3$ is the z-intercept

$x=z=0 \Rightarrow y=3$ " " y-intercept

$y=z=0 \Rightarrow x=3$ " " x-intercept

3) $z=0$, xy-plane, Gray plane

$x=y=0 \Rightarrow z=0$ is the z-intercept

$y=0=z=0 \Rightarrow$ any $x\in\mathbb{R}$ is x-intercept, all points of x-axis are x-intercepts

$x=z=0 \Rightarrow$ any $y\in\mathbb{R}$ is y-intercept, all points of y-axis are y-intercepts.

Equation of the plane passing through P_0, P_1, P_2 :

if $P_0=(x_0,y_0,z_0)$, $P_1=(x_1,y_1,z_1)$, $P_2=(x_2,y_2,z_2)$

$P_1 \xleftarrow{\vec{u}_1} P_0 \xrightarrow{\vec{u}_2} P_2$ $\quad \left.\begin{array}{l}\vec{u}_1=\overrightarrow{P_0P_1}\\ \vec{u}_2=\overrightarrow{P_0P_2}\end{array}\right\} \Rightarrow \vec{n}=\vec{u}_1\times\vec{u}_2$

Then use any of the points together with $\vec{n}$ to get the equation for the plane.

($\vec{u}_1 /\!/ \vec{u}_2 \Leftrightarrow \vec{n}=\vec{0}$ no plane)
P_0, P_1, P_2 are all on the same line.

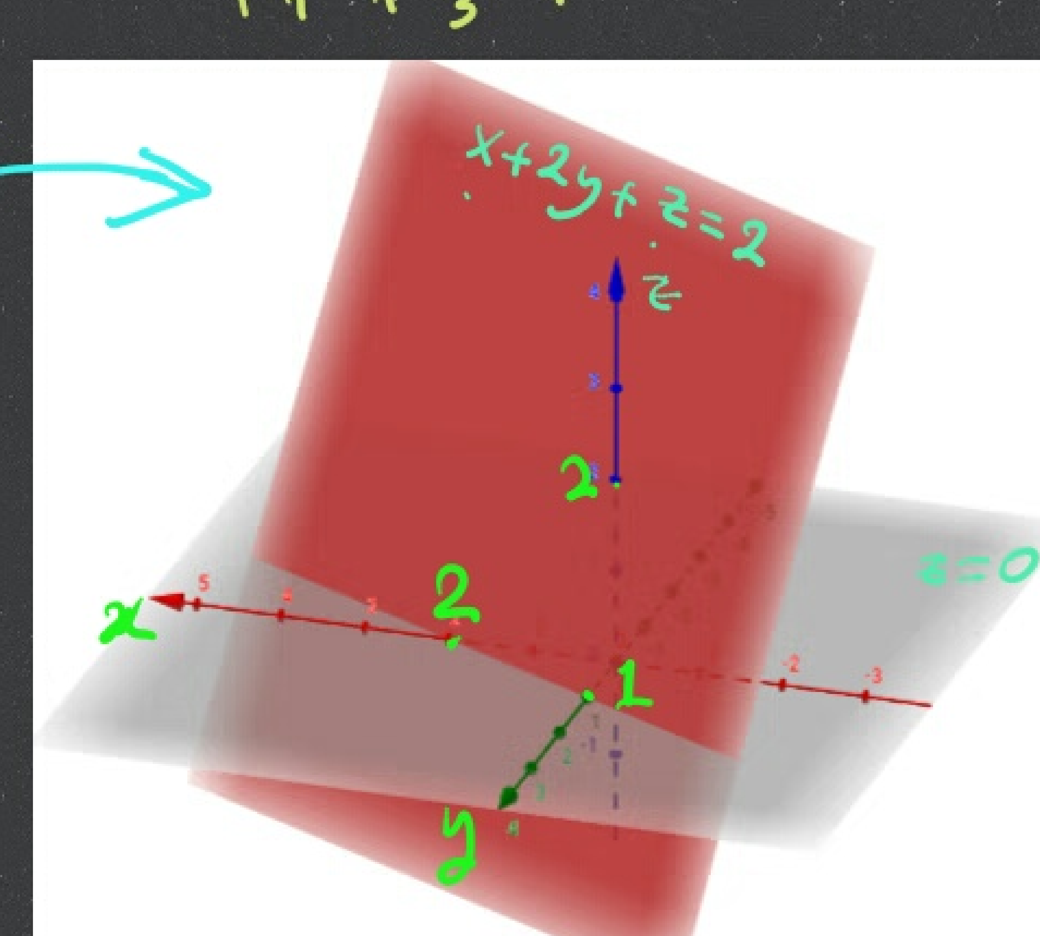
Example: Let $P_0=(0,1,0)$, $P_1=(1,1,1)$, $P_2=(1,2,3)$

What is the equation of the plane containing P_0, P_1, P_2?

$\vec{u}_1=\overrightarrow{P_0P_1}=\underset{P_1-P_0}{(1,0,1)}$, $\vec{u}_2=\overrightarrow{P_0P_2}=\underset{P_2-P_0}{(1,1,3)} \Rightarrow \vec{n}=\vec{u}_1\times\vec{u}_2=\begin{vmatrix}\vec{i}&\vec{j}&\vec{k}\\1&0&1\\1&1&3\end{vmatrix}=-\vec{i}-2\vec{j}-\vec{k}$

$\Rightarrow -(x-0)-2(y-1)-(z-0)=0$

$-x-2y-z=-2 \Leftrightarrow x+2y+z=2$



Remarks:

- Two planes are parallel $\Leftrightarrow \vec{n}_1 \parallel \vec{n}_2$.

- If two planes are not parallel, they intersect in a line.

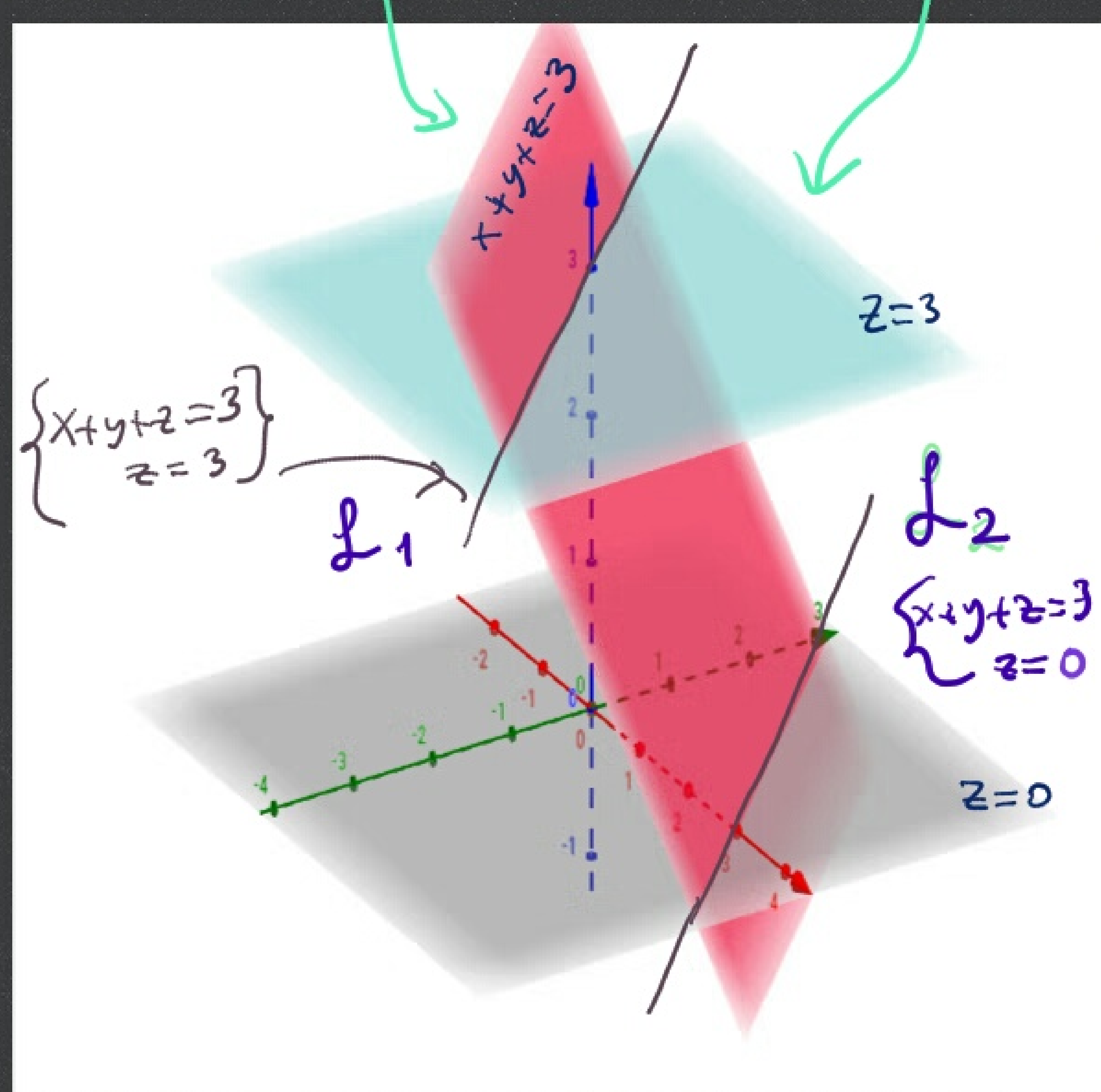
  $x+y+z=3$ intersected with $z=3$ is a line

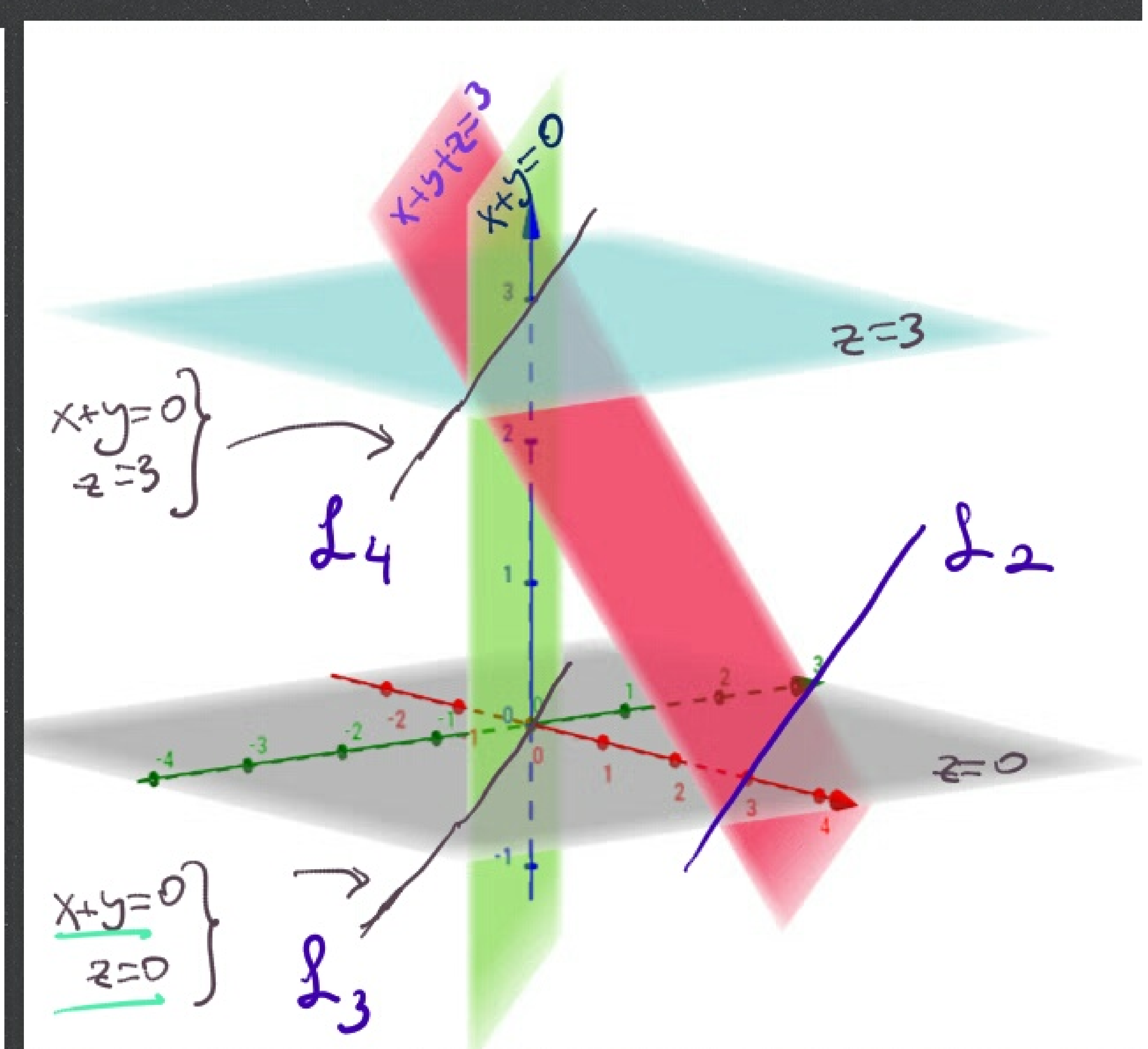
$\mathcal{L}_1$

$$\begin{array}{r} x+y+z=3 \\ -(\quad z=3) \\ \hline x+y=0 \end{array}$$

Question: Is $x+y=0$ an equation for $\mathcal{L}_1$?

No!





It is an equation for the projection of $\mathcal{L}_1$ on the xy-plane as $x+y=0$ describes a line in $\mathbb{R}^2$. $\mathcal{L}_3$ is the projection of $\mathcal{L}_1$ on the xy-plane.

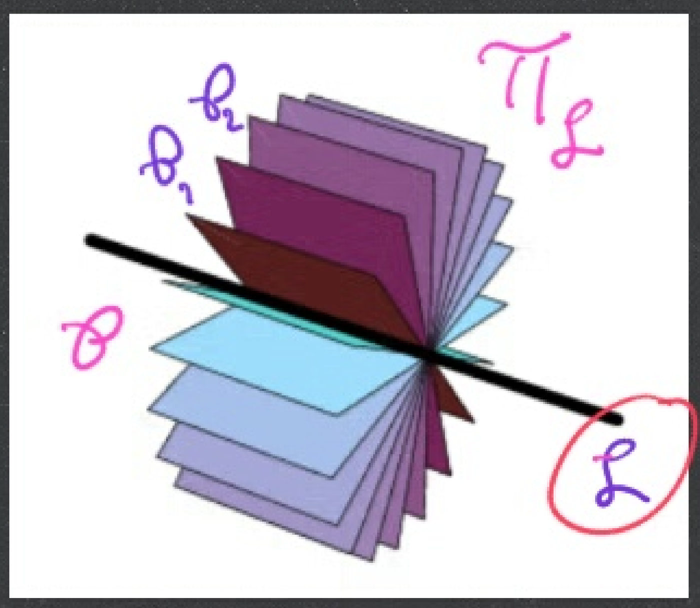
$x+y=0$ describes a plane in $\mathbb{R}^3$ vertical, green plane.

Also, $x+y=3$ desribes the line $\mathcal{L}_2$ which is on the xy-plane. it coincides with the intersection of the plane $x+y=3$ in $\mathbb{R}^3$, which is a plane parallel to the green plane $x+y=0$, with the xy-plane $(z=0)$.

A Pencil of planes is a set of planes intersecting in a straight line.

Let $\mathcal{P}_1 : A_1x+B_1y+C_1z-D_1=0$, $\mathcal{P}_2 : A_2x+B_2y+C_2z-D_2=0$ be two non-parallel planes whose intersection is the line $\mathcal{L}$.

Let $\Pi_{\mathcal{L}}$ be the pencil of planes intersecting in $\mathcal{L}$.



Fact: If $\mathcal{P} \in \Pi_{\mathcal{L}}$ and $\mathcal{P} \neq \mathcal{P}_2$, then there is $\lambda \in \mathbb{R}$ such that $\mathcal{P} : A_1x+B_1y+C_1z-D_1+\lambda(A_2x+B_2y+C_2z-D_2)=0$

EXAMPLE 4 Find an equation of the plane $\mathcal{P}$ passing through the line of intersection of the two planes

$\mathcal{P}_1 \cap \mathcal{P}_2 = \mathcal{L} \subset \mathcal{P}$

$\mathcal{P}_1, \mathcal{P}_2, \mathcal{P} \in \Pi_{\mathcal{L}}$

$\mathcal{P}_1$: $x + y - 2z = 6$ and $\mathcal{P}_2$: $2x - y + z = 2$

and also passing through the point $(-2, 0, 1)$.

Solution By the fact above there is $\lambda \in \mathbb{R}$ s.t.

$\mathcal{P}$: $x + y - 2z - 6 + \lambda(2x - y + z - 2) = 0$

This plane passes through the point $(-2, 0, 1)$ if $-2 - 2 - 6 + \lambda(-4 + 1 - 2) = 0$, then $\lambda = -2$. The equation of the required plane therefore simplifies to $3x - 3y + 4z + 2 = 0$.

Proof of Fact: For $\lambda \in \mathbb{R}$, $A_1x+B_1y+C_1z-D_1+\lambda(A_2x+B_2y+C_2z-D_2)=0$ defines a plane Q_λ: $(A_1+\lambda A_2)x+(B_1+\lambda B_2)y+(C_1+\lambda C_2)z-(D_1+\lambda D_2)=0$ (*)

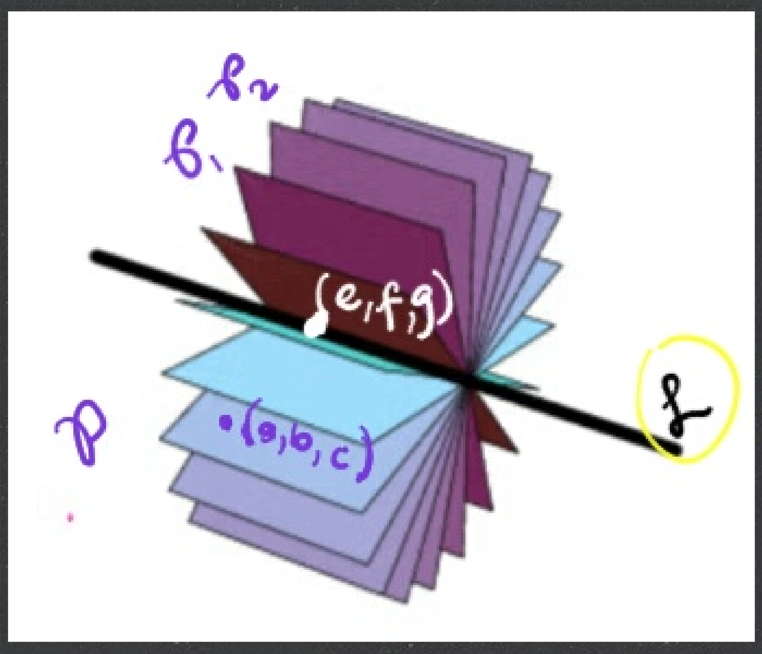
Is $Q_\lambda \in \Pi_{\mathcal{L}}$?

If $(e,f,g) \in \mathcal{L}$, then $A_1e+B_1f+C_1g-D_1=0$ because $(e,f,g) \in \mathcal{P}_1$

$A_2e+B_2f+C_2g-D_2=0$ " $(e,f,g) \in \mathcal{P}_2$

Then (e,f,g) satisfies $A_1e+B_1f+C_1g-D_1+\lambda(A_2e+B_2f+C_2g-D_2)=0$

So, $(e,f,g) \in Q_\lambda$. Therefore $\mathcal{L} \subset Q_\lambda \Rightarrow Q_\lambda \in \Pi_{\mathcal{L}}$.



Now we will show that if $\mathcal{P} : Ax+By+Cz-D=0$ is in $\Pi_{\mathcal{L}}$ and $\mathcal{P} \neq \mathcal{P}_2$, there is $\lambda \in \mathbb{R}$ such that $\mathcal{P} = Q_\lambda$.

Since $\mathcal{P} \neq \mathcal{P}_2$, there is $(a,b,c) \in \mathcal{P}$, $(a,b,c) \notin \mathcal{P}_2$.

Then $Aa+Bb+Cc-D=0$, but $A_2a+B_2b+C_2c-D_2 \neq 0$.

Then $Aa+Bb+Cc-D=0=A_1a+B_1b+C_1c-D_1-\left(\frac{A_1a+B_1b+C_1c-D_1}{A_2a+B_2b+C_2c-D_2}\right)\cdot(A_2a+B_2b+C_2c-D_2)=0$

Also, for $(e,f,g) \in \mathcal{L} \subset \mathcal{P} \cap \mathcal{P}_1 \cap \mathcal{P}_2$, $\lambda \in \mathbb{R}$, $\underbrace{Ae+Bf+Cg-D}_{\overset{\|}{0}\ (e,f,g)\in\mathcal{P}} = \underbrace{A_1e+B_1f+C_1g-D_1}_{(e,f,g)\in\mathcal{P}_1 \to \overset{\|}{0}} + \lambda\cdot\underbrace{(A_2e+B_2f+C_2g-D_2)}_{(e,f,g)\in\mathcal{P}_2 \to \overset{\|}{0}} = 0$

Therefore for any $(x,y,z) \in \mathcal{P}$, $\underset{\in \mathcal{L}}{(x,y,z)=(e,f,g)}$ or $(x,y,z)=(a,b,c) \notin \mathcal{L}$ in any case

Then $\mathcal{P}$: $0=Ax+By+Cz-D=A_1x+B_1y+C_1z-D_1-\left(\frac{A_1a+B_1b+C_1c-D_1}{A_2a+B_2b+C_2c-D_2}\right)(A_2x+B_2y+C_2z-D_2)=0$

That is $\mathcal{P} = Q_\lambda = Q_{-\left(\frac{A_1a+B_1b+C_1c-D_1}{A_2a+B_2b+C_2c-D_2}\right)}$.

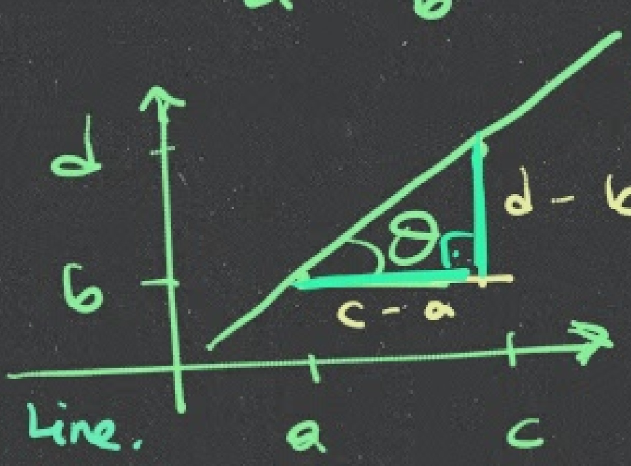
Lines in 3-Space, $\mathbb{R}^3$.

Lines in $\mathbb{R}^2$ are of the form $ax+by=c$ or $\frac{x}{a}+\frac{y}{b}=1$ when $a\neq 0, b\neq 0$

A vertical line in $\mathbb{R}^2$ is of the form $x=a$,

if (a,b), (c,d) are on a non-vertical line $\mathcal{L}$

then $y-b=\left(\frac{d-b}{c-a}\right)(x-a)$ is the equation of $\mathcal{L}$.

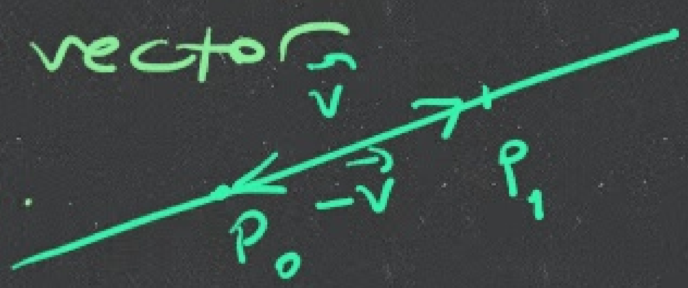


- Recall that two distinct points in $\mathbb{R}^n$ determine a unique line.

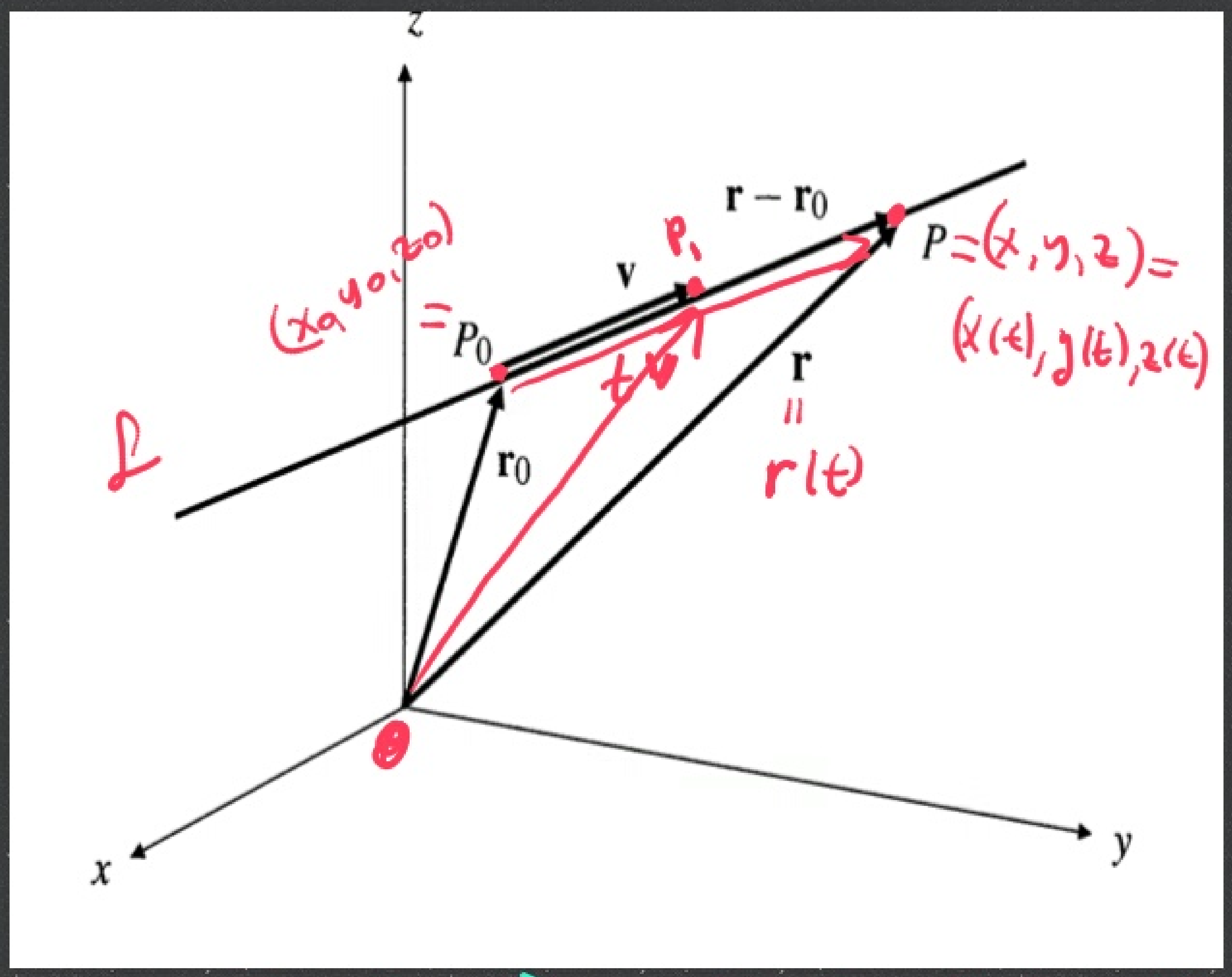
Let's obtain the vector parametric equation of a line in $\mathbb{R}^3$.

- If P_0, P_1 are two distinct points on $\mathcal{L}$, then the vector $\vec{v}$

$\vec{v}=\overrightarrow{P_0P_1}$, or $-\vec{v}=\overrightarrow{P_1P_0}$ determines a direction vector of $\mathcal{L}$.



A direction vector $\vec{v}$ and a point $P_0=(x_0,y_0,z_0)$ on $\mathcal{L}$ is sufficient to describe the points of $\mathcal{L}$ as the tips (heads) of the vectors of the form $\vec{r}(t)=\vec{r}_0+t\vec{v}$, $t\in\mathbb{R}$, $\vec{r}_0=\overrightarrow{OP_0}$, $\vec{v}=\overrightarrow{P_0P_1}$,



$\vec{r}(0)=\vec{r}_0=\overrightarrow{OP_0}$

$\vec{r}(1)=\vec{r}_0+\overrightarrow{P_0P_1}=\overrightarrow{OP_0}+\overrightarrow{P_0P_1}=\overrightarrow{OP_1}$

tip of $\vec{r}(t)$ is on $\mathcal{L}$.

Write $\vec{r}(t)=x(t)\vec{i}+y(t)\vec{j}+z(t)\vec{k}$

$(x(t), y(t), z(t))$ is the tip of the vector $\vec{r}$ with tail at $(0,0,0)$.

To make it more explicit, let's take $\vec{v}=a\vec{i}+b\vec{j}+c\vec{k}$ and $P_0=(x_0,y_0,z_0)$ on $\mathcal{L}$, then $\vec{r}_0=\overrightarrow{OP_0}=x_0\vec{i}+y_0\vec{j}+z_0\vec{k}$

and $t\vec{v}=at\vec{i}+tb\vec{j}+tc\vec{k}$

then $\vec{r}(t)=\vec{r}_0+t\vec{v}=\underbrace{(x_0+ta)}_{x(t)}\vec{i}+\underbrace{(y_0+tb)}_{y(t)}\vec{j}+\underbrace{(z_0+tc)}_{z(t)}\vec{k}$ vector parametric eqn. of $\mathcal{L}$

t is the parameter $t\in\mathbb{R}$.

$$\left.\begin{aligned} x&=x(t)=x_0+at\\ y&=y(t)=y_0+bt\\ z&=z(t)=z_0+ct\\ &t\in\mathbb{R}\end{aligned}\right\}$$

if $abc\neq 0$ $\Longrightarrow$ $t=\frac{x-x_0}{a}=\frac{y-y_0}{b}=\frac{z-z_0}{c}$. symmetric equations . scalar parametric equations of $\mathcal{L}$

if $ab\neq 0, c=0$ $\Longrightarrow$ $t=\frac{x-x_0}{a}=\frac{y-y_0}{b}$ and $z=z_0$

$a\neq 0, b=0=c$ $\Longrightarrow$ $t=\frac{x-x_0}{a}$ and $y=y_0$ and $z=z_0$

Also, as a function $\mathcal{L}:\mathbb{R}\to\mathbb{R}^3$ can be written as

$$\mathcal{L}(t)=(x(t), y(t), z(t))=(x_0+at, y_0+bt, z_0+ct) \text{ for } t\in\mathbb{R}$$

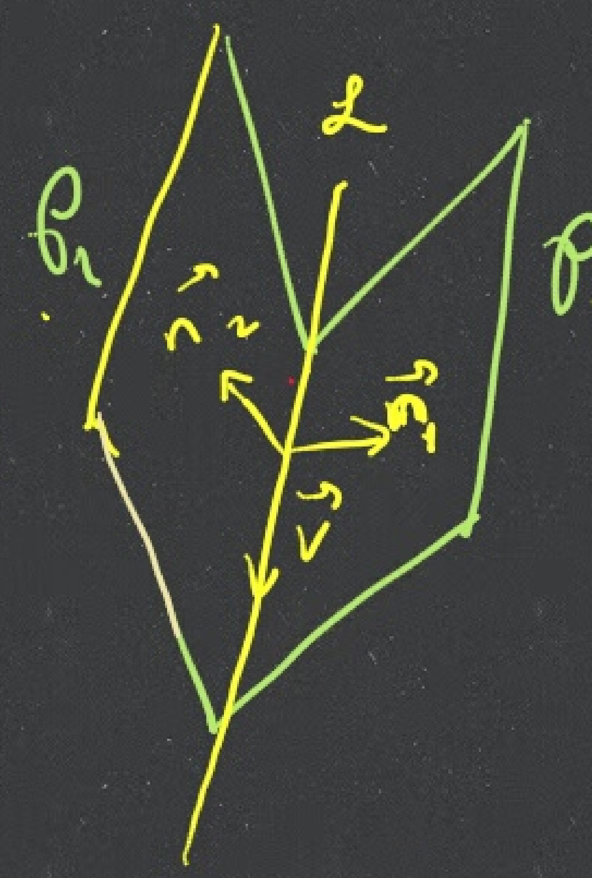
we can recover P_0 & $\vec{v}$ from $\mathcal{L}(t)$ as follows:

$\mathcal{L}(0)=(x_0,y_0,z_0)=P_0$, $\mathcal{L}(1)=(x_0+a, y_0+b, z_0+c)\Rightarrow \vec{v}=\overrightarrow{\mathcal{L}(0)\mathcal{L}(1)}=a\vec{i}+b\vec{j}+c\vec{k}$.

Example ① Find the vector parametric and standard equation for the line $\mathcal{L}$ which is the intersection of the planes $\mathcal{P}_1 : 1 \cdot x + 1y - 2z = 6$, and $\mathcal{P}_2 : 2x \ y + z = 2$.

$$\Downarrow \qquad\qquad\qquad\qquad \Downarrow$$

$$\vec{n}_1 = 1\vec{i} + 1\vec{j} - 2\vec{k} \qquad\qquad \vec{n}_2 = 2\vec{i} - \vec{j} + \vec{k}$$



$$\vec{v} = \vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & -2 \\ 2 & -1 & 1 \end{vmatrix} = -\vec{i} - 5\vec{j} - 3\vec{k}$$

we may take $-\vec{v}$ as well
$$-\vec{v} = \vec{i} + 5\vec{j} + 3\vec{k}$$

We need a point on $\mathcal{L}$;

any point (a, b, c) satisfying both equations is fine.

2 eqn. 3 variables

$$\begin{aligned} a + b - 2c &= 6 \\ + \quad 2a - b + c &= 2 \\ \hline 3a - c &= 8 \end{aligned} \Rightarrow \quad 3a = 8 + c \quad ; \quad \text{let } \underset{\text{arbitrary}}{c = -2} \Rightarrow a = 2 \Rightarrow b = 6 - a + 2c$$
$$= 6 - 2 - 4 = 0$$

$(x_0, y_0, z_0) = (2, 0, -2)$ is on $\mathcal{P}_1$ & $\mathcal{P}_2$ so it is on $\mathcal{L}$.

$$\vec{r}(t) = 2\vec{i} - 2\vec{k} + t(\vec{i} + 5\vec{j} + 3\vec{k})$$
$$= \underset{x(t)}{(2+t)}\vec{i} + \underset{y(t)}{5t}\vec{j} + \underset{z(t)}{(-2+3t)}\vec{k}$$

Vector parametric eqn. for $\mathcal{L}$.

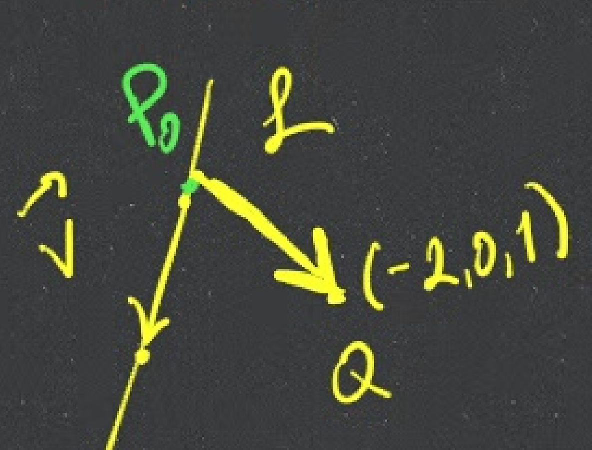
$$x(t) = x = 2 + t$$
$$y(t) = y = 5t$$
$$z(t) = z = -2 + 3t$$

$$\Rightarrow \quad t = \frac{x-2}{1} = \frac{y}{5} = \frac{z+2}{3}$$

symmetric eqns. is the standard form eqn.s for $\mathcal{L}$

- $\mathcal{L}(t) = (2+t, 5t, -2+3t)$ as a function from $\mathbb{R}$ to $\mathbb{R}^3$.

② Find the plane passing through above $\mathcal{L}$ and the point $(-2, 0, 1) = Q$.



We need $\vec{n}$ for the plane.

Take P_0 on $\mathcal{L}$, say, $P_0 = \mathcal{L}(0) = (2, 0, -2) (\neq Q)$

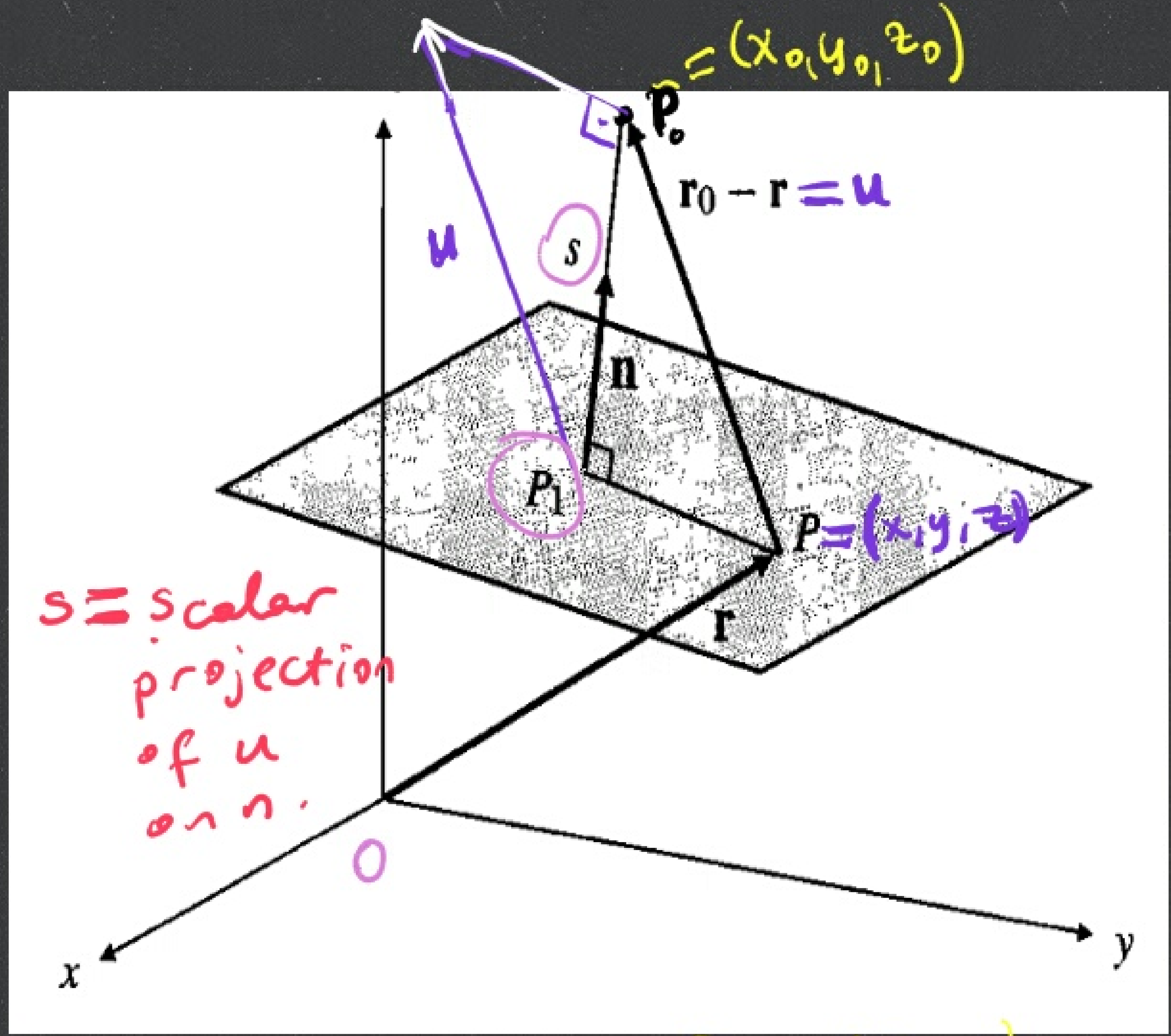
then $\vec{n} = \vec{v} \times \vec{P_0Q}$, where $\vec{P_0Q} = -4\vec{i} + 0 \cdot \vec{j} + 3\vec{k}$
$$\vec{v} = \vec{i} + 5\vec{j} + 3\vec{k}$$

then $$\vec{n} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 5 & 3 \\ -4 & 0 & 3 \end{vmatrix} = \underset{A}{15}\vec{i} - \underset{B}{15}\vec{j} + \underset{C}{20}\vec{k}$$

$$\mathcal{P} : 0 = A(x - x_0) + B(y - y_0) + C(z - z_0), \qquad (x_0, y_0, z_0) = P_0 = (2, 0, -2)$$

$$0 = 15(x-2) - 15(y) + 20(z+2)$$
$$= 15x - 30 - 15y + 20z + 40 = \boxed{15x - 15y + 20z + 10 = 0} \quad \boxed{3x - 3y + 4z + 2 = 0}$$

- What is the distance formula from a point P_0 to a plane $\mathcal{P}$.
  Distance from P_0 to $\mathcal{P}$ is S; length of the projection vector of $\vec{u}$ on $\vec{n}$.
  Where $\vec{u}=\overrightarrow{PP_0}$, $\vec{n}$ is the normal of the plane $\mathcal{P}$. $\Rightarrow S=\dfrac{\vec{u}\cdot\vec{n}}{|\vec{n}|}$



$\mathcal{P}: Ax+By+Cz=D \Rightarrow \vec{n}=A\vec{i}+B\vec{j}+C\vec{k}$

$\vec{u}=\overrightarrow{PP_0}=(x_0-x)\vec{i}+(y_0-y)\vec{j}+(z_0-z)\vec{k}$

$|\vec{n}|=\sqrt{\vec{n}\cdot\vec{n}}=\sqrt{A^2+B^2+C^2}$

$$\Rightarrow S=\frac{\vec{n}\cdot\vec{u}}{|\vec{n}|}=\frac{|A(x_0-x)+B(y_0-y)+C(z_0-z)|}{\sqrt{A^2+B^2+C^2}}$$ no P_1 !

$$=\frac{|Ax_0+By_0+Cz_0-\overbrace{(Ax+By+Cz)}^{=D}|}{\sqrt{A^2+B^2+C^2}}$$

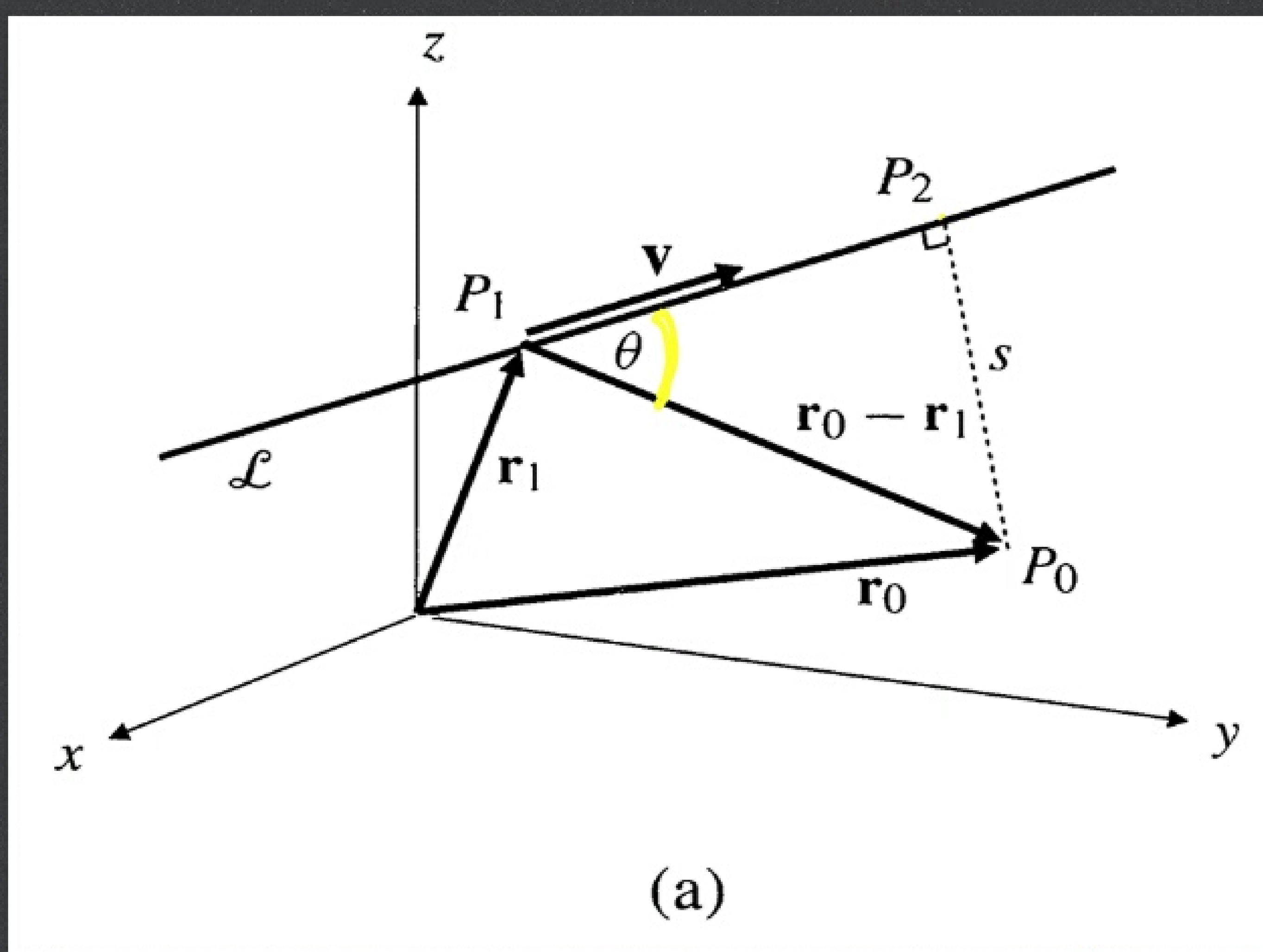
$$=\frac{|Ax_0+By_0+Cz_0-D|}{\sqrt{A^2+B^2+C^2}} = \left\{\text{Distance from } \underset{\substack{\| \\ }}{\overset{(x_0,y_0,z_0)}{P_0}} \text{ to the plane } \mathcal{P}: Ax+By+Cz-D=0\right\}$$

Recall that $\vec{u}_{\vec{n}}=\underbrace{\left(\dfrac{\vec{u}\cdot\vec{n}}{|\vec{n}|}\right)}_{s}\underbrace{\dfrac{\vec{n}}{|\vec{n}|}}_{\text{unit vector along } \vec{n}}$

- What is the distance formula from a point to a line?

Let $\mathcal{L}$ be a line with direction vector $\vec{v}$ and P_0 be a point not on $\mathcal{L}$.

As seen in the picture below S is the required distance, where P_2 is the point on $\mathcal{L}$ which is closest to P_0,



(a)

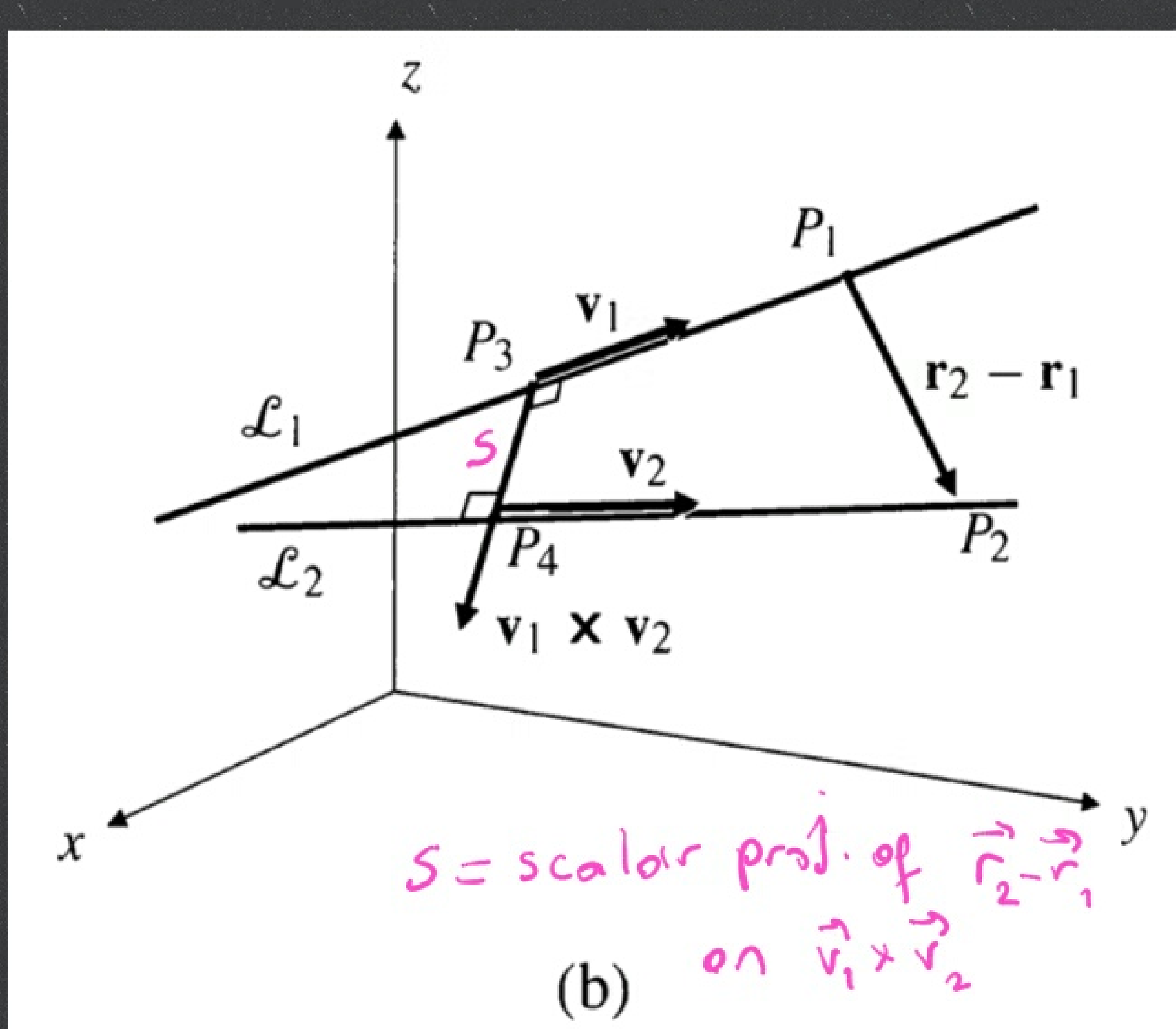
$$S = |\overrightarrow{P_2P_0}| = |\overrightarrow{P_1P_0}| \cdot \sin(\theta) = |\vec{r}_0 - \vec{r}_1| \cdot \sin(\theta) = \frac{|\vec{r}_0 - \vec{r}_1|\,|\vec{v}| \cdot \sin\theta}{|\vec{v}|}$$

$$= \frac{|(\vec{r}_0 - \vec{r}_1) \times \vec{v}|}{|\vec{v}|}$$ length of the cross product

Note that P_2 is not in the formula, only a random point P_1 on $\mathcal{L}$ is sufficient to compute S.

- What is the formula for distance between two lines?

Distance between to lines $\mathcal{L}_1$ and $\mathcal{L}_2$ with direction vectors $\vec{v}_1, \vec{v}_2$ respectively is S which is the length of the projection (scalar projection) of $\vec{r}_2 - \vec{r}_1$ on $\vec{v}_1 \times \vec{v}_2$;



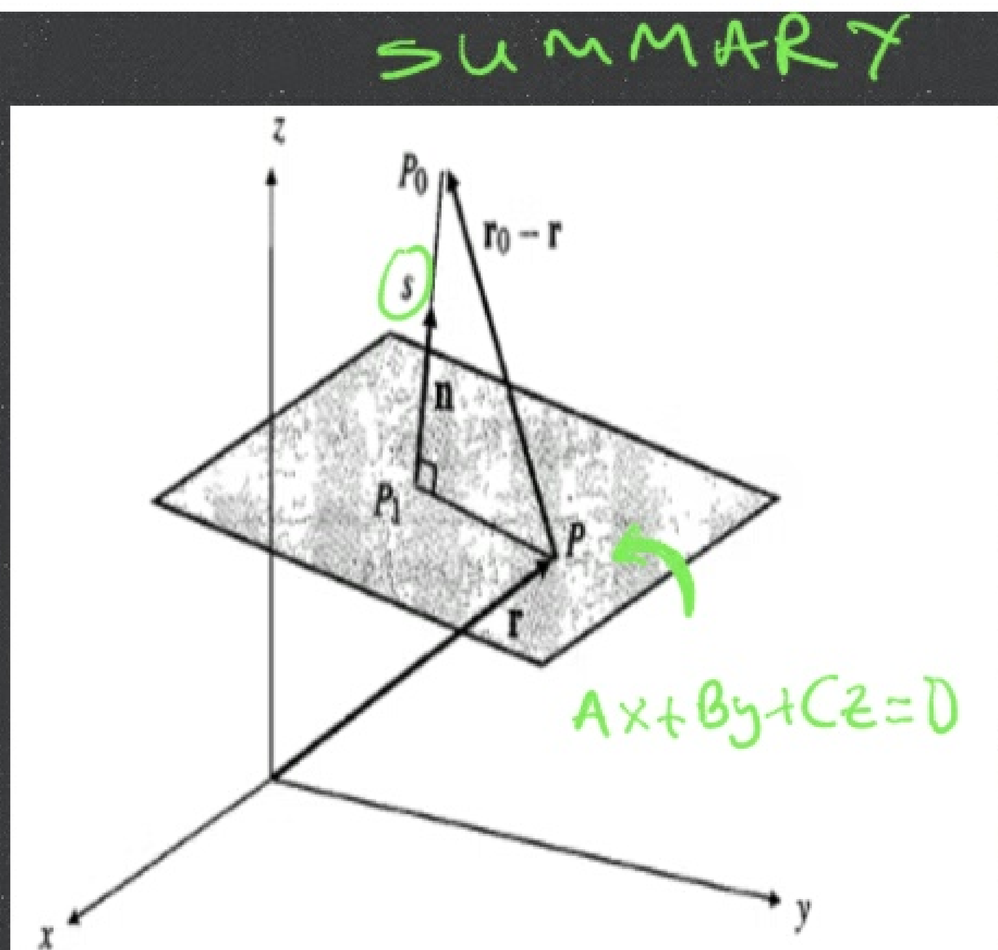
(b)

$$S = \frac{|(\vec{r}_2 - \vec{r}_1) \cdot (\vec{v}_1 \times \vec{v}_2)|}{|\vec{v}_1 \times \vec{v}_2|}$$

note that no P_3, P_4

which requires P_1 on $\mathcal{L}_1$, P_2 on $\mathcal{L}_2$ so that $\vec{r}_2 = \overrightarrow{OP_2}$, $\vec{r}_1 = \overrightarrow{OP_1}$ & $\vec{v}_1, \vec{v}_2$ are the direction vectors of $\mathcal{L}_1, \mathcal{L}_2$ respectively.

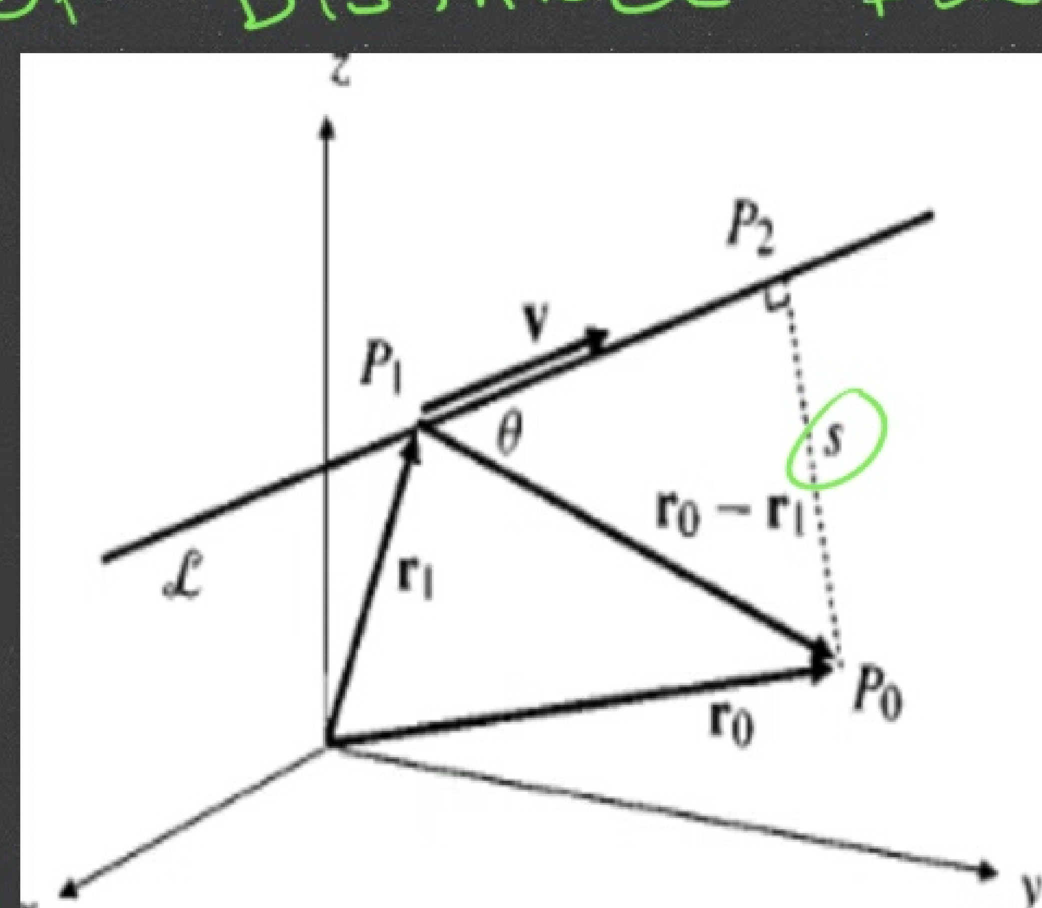
SUMMARY OF DISTANCE FORMULAS



$$S = \frac{|Ax_0 + By_0 + Cz_0 - D|}{\sqrt{A^2 + B^2 + C^2}}$$

where $P_0 = (x_0, y_0, z_0)$

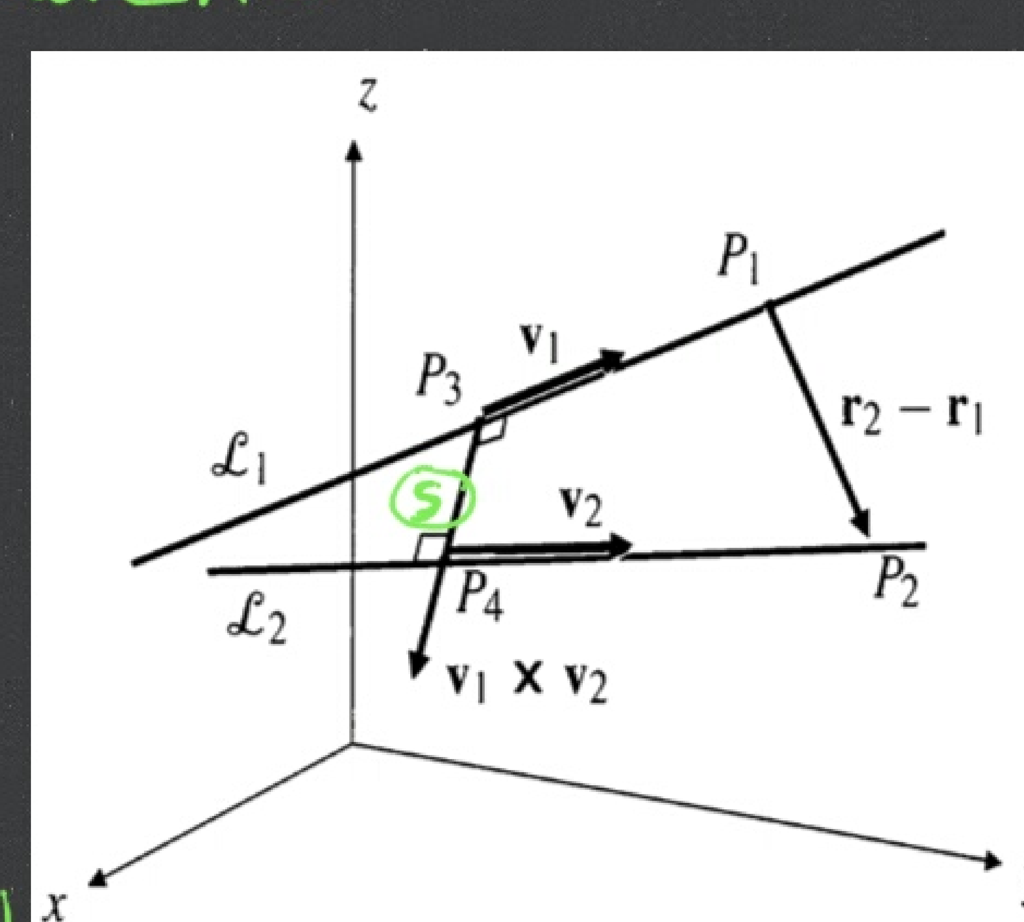
(by projection of $\vec{r}_0 - \vec{r}$ on $\vec{n}$)



$$S = \frac{|(\vec{r}_0 - \vec{r}_1) \times \vec{v}|}{|\vec{v}|}$$ ← length

where $\vec{r}_0 - \vec{r}_1 = \overrightarrow{P_1P_0}$

P_1 is on $\mathcal{L}$, P_0 is not on $\mathcal{L}$



$$S = \frac{|(\vec{r}_2 - \vec{r}_1) \cdot (\vec{v}_1 \times \vec{v}_2)|}{|\vec{v}_1 \times \vec{v}_2|}$$

where $\vec{r}_2 - \vec{r}_1 = \overrightarrow{P_1P_2}$

(by projection of $\vec{r}_2 - \vec{r}_1$ on $\vec{v}_1 \times \vec{v}_2$)