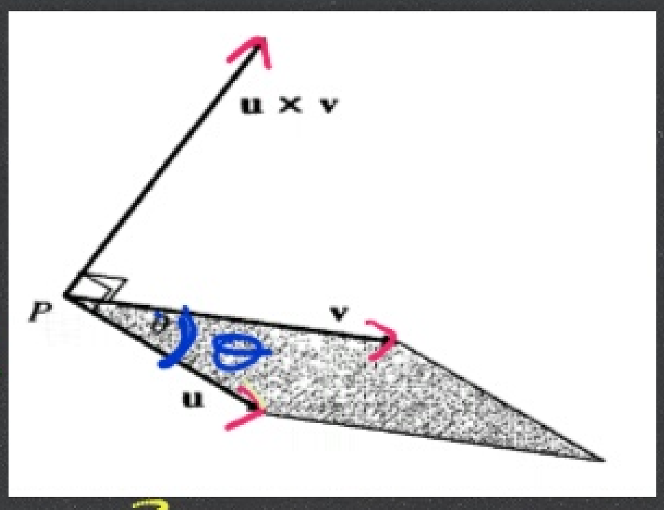
10.3. The Cross Product in 3-space



Definition: The cross product $\vec{u}\times\vec{v}$ of any two vectors $\vec{u},\vec{v}$ in $\mathbb{R}^3$ is the unique vector satisfying the three conditions below:

(P1) $\vec{u}\times\vec{v}$ is perpendicular to both $\vec{u}$ & $\vec{v}$ $\equiv$ $\vec{u}\cdot(\vec{u}\times\vec{v})=0=\vec{v}\cdot(\vec{u}\times\vec{v})$

(P2) $|\vec{u}\times\vec{v}|=|\vec{u}||\vec{v}|\cdot\sin(\theta)$ where θ is the angle between $\vec{u}$ & $\vec{v}$, $0\le\theta\le\pi$.

(P3) $\vec{u}, \vec{v}, \vec{u}\times\vec{v}$ forms right-handed triad (as $\vec{i},\vec{j},\vec{k}$).

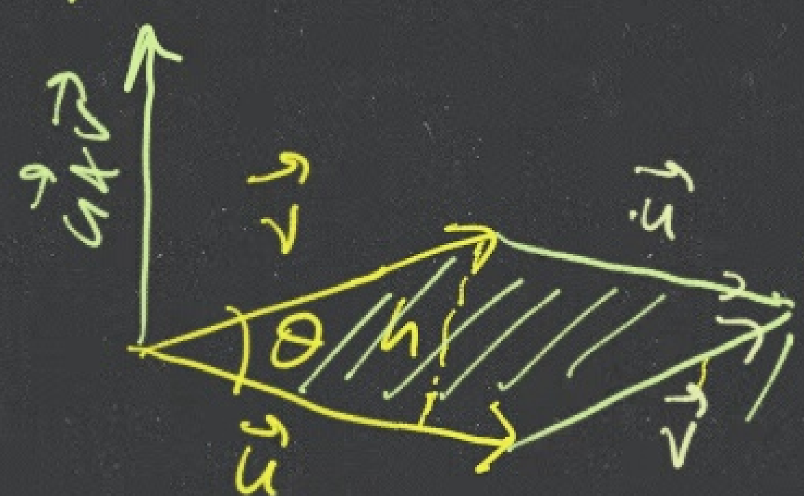
OBSERVATIONS:

(1) $\vec{u}/\!/\vec{v} \Rightarrow \vec{u}=t\vec{v}$ for some $t\in\mathbb{R}$ $\overset{\text{by P2}}{\Rightarrow}$ $|\underbrace{(t\vec{v})}_{\vec{u}}\times\vec{v}|=|t||\vec{v}|^2\underbrace{\sin(\theta)}_{0}=0$ because $\theta=0$ or $\theta=\pi$, $\boxed{\text{so } \vec{u}\times\vec{v}=\vec{0}}$ $\boxed{\vec{u}=t\vec{v}}$

$\vec{v}$ rectangle $\vec{u}$

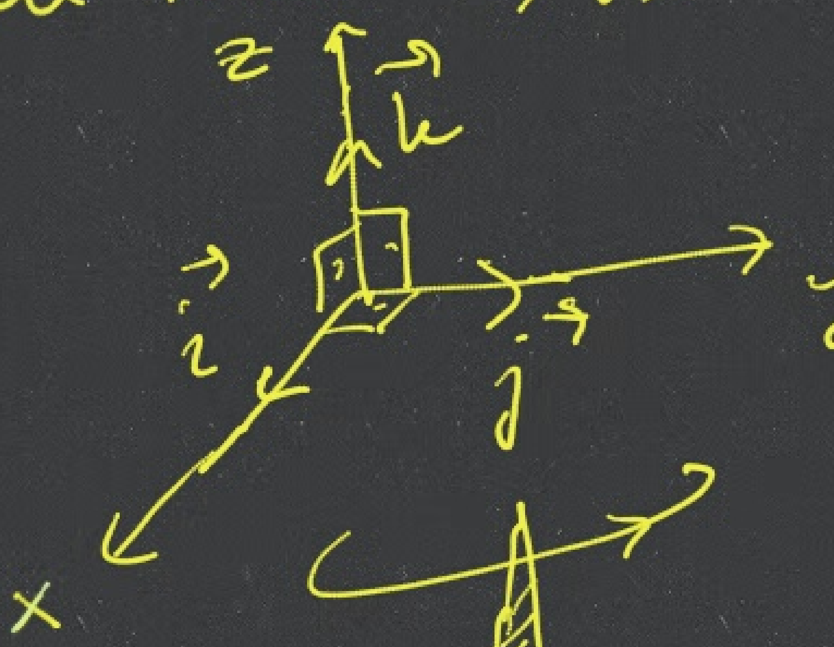
(2) $|\vec{u}\times\vec{v}|=|\vec{u}|\cdot|\vec{v}|\cdot\overbrace{\sin(\theta)}^{\le 1}\le|\vec{u}||\vec{v}|$ and $|\vec{u}\times\vec{v}|=|\vec{u}||\vec{v}|$ if $\vec{u}\perp\vec{v}$.

(3)

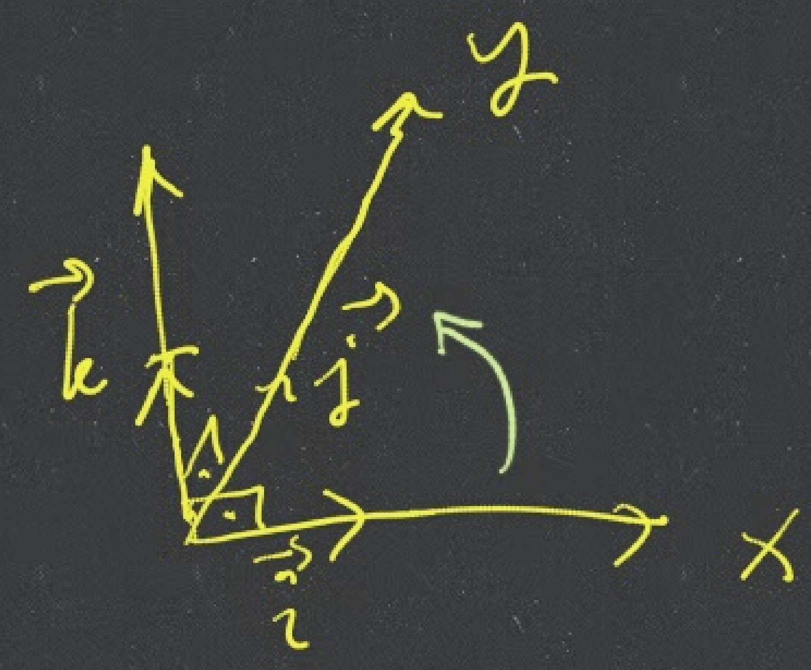


$h=|\vec{v}|\cdot\sin(\theta)$ $\Rightarrow$ Area $=|\vec{u}|\cdot h=|\vec{u}||\vec{v}|\sin\theta$ $=|\vec{u}\times\vec{v}|$.

(4) Recall that $\vec{i},\vec{j},\vec{k}$ form a right-handed triad (right handed orientation (10.1))



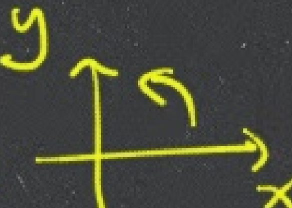
$\equiv$



$\vec{k}$ satisfies P1, P2, P3 for $\vec{u}=\vec{i}$, $\vec{v}=\vec{j}$ $\Rightarrow \boxed{\vec{k}=\vec{i}\times\vec{j}}$

(10.1) right handed orientation

Screw moves upward when twisted counterclockwise



How to compute $\vec{u}\times\vec{v}$?

[Notation: $\begin{vmatrix} a & b \\ c & d \end{vmatrix}=ad-bc$ determinant]

Let $\vec{u}=u_1\vec{i}+u_2\vec{j}+u_3\vec{k}$ and $\vec{v}=v_1\vec{i}+v_2\vec{j}+v_3\vec{k}$, then

$$\vec{u}\times\vec{v}=(u_2v_3-u_3v_2)\vec{i}-(u_1v_3-u_3v_1)\vec{j}+(u_1v_2-u_2v_1)\vec{k}.$$

$$=\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}\vec{i}-\begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}\vec{j}+\begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}\vec{k}$$

$$\begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$$\left(\vec u = u_1\vec i + u_2\vec j + u_3\vec k\,,\quad \vec v = v_1\vec i + v_2\vec j + v_3\vec k\right) \Rightarrow$$

$$\left(\vec u\times\vec v = (u_2v_3-u_3v_2)\vec i - (u_1v_3-u_3v_1)\vec j + (u_1v_2-u_2v_1)\vec k\right)$$

Examples:

① $\vec u = \vec j$, $\vec v = \vec i$ $\Rightarrow$ $\vec j\times\vec i = 0\vec i + 0\vec j + (0-1)\vec k = -\vec k$

$u_1 = u_3 = 0$, $u_2 = 1$; $v_2 = v_3 = 0$, $v_1 = 1$

$(-\circlearrowleft)$ $\left(\vec i \to \vec j \to \vec k \to \vec i\right)$

$\vec i\times\vec j = \vec k$, $\vec j\times\vec k = \vec i$, $\vec k\times\vec i = \vec j$; $\vec j\times\vec i = -\vec k$, $\vec k\times\vec j = -\vec i$, $\vec i\times\vec k = -\vec j$

② $\boxed{\vec u = \vec j - \vec k}$, $\boxed{\vec v = 2\vec i + \vec k}$ $\Rightarrow \vec u\times\vec v = \vec i - 2\vec j - 2\vec k$

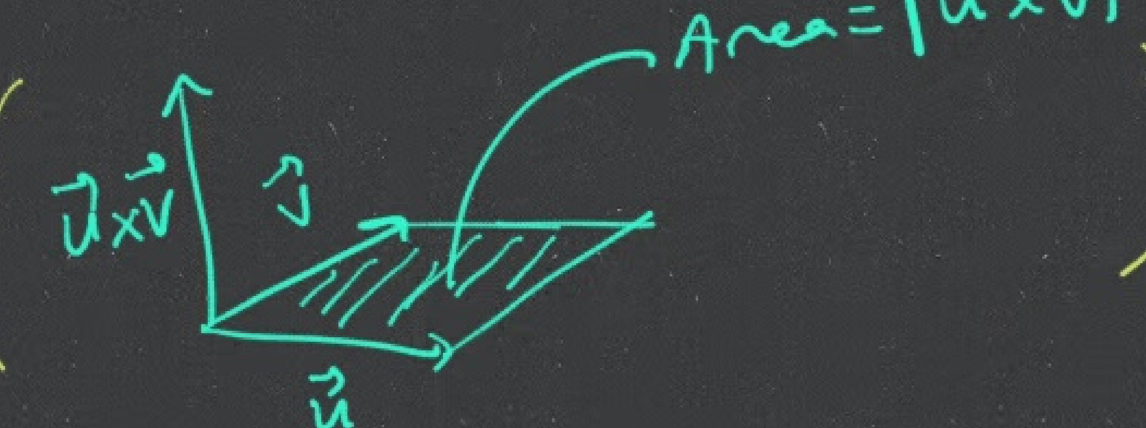
$u_1 = 0,\ u_2 = 1,\ u_3 = -1$; $v_1 = 2,\ v_2 = 0,\ v_3 = 1$

$\Rightarrow$ the area of the parallelogram with sides $\vec u$ & $\vec v$ is $|\vec u\times\vec v| = \sqrt{1+4+4} = 3$

$3 = |\vec u\times\vec v| \le |\vec u||\vec v| = \sqrt2\sqrt5 = \sqrt{10}$.

Properties of cross product.

For $\vec u, \vec v, \vec w$ in $\mathbb{R}^3$, $t\in\mathbb{R}$;



1) $\vec u\times\vec u = 0$

(2) $\vec u\times\vec v = -\vec v\times\vec u$

3) $(\vec u+\vec v)\times\vec w = \vec u\times\vec w + \vec v\times\vec w$

(4) $\vec u\times(\vec v+\vec w) = \vec u\times\vec v + \vec u\times\vec w$

(5) $(t\vec u)\times\vec v = \vec u\times(t\vec v) = t(\vec u\times\vec v)$

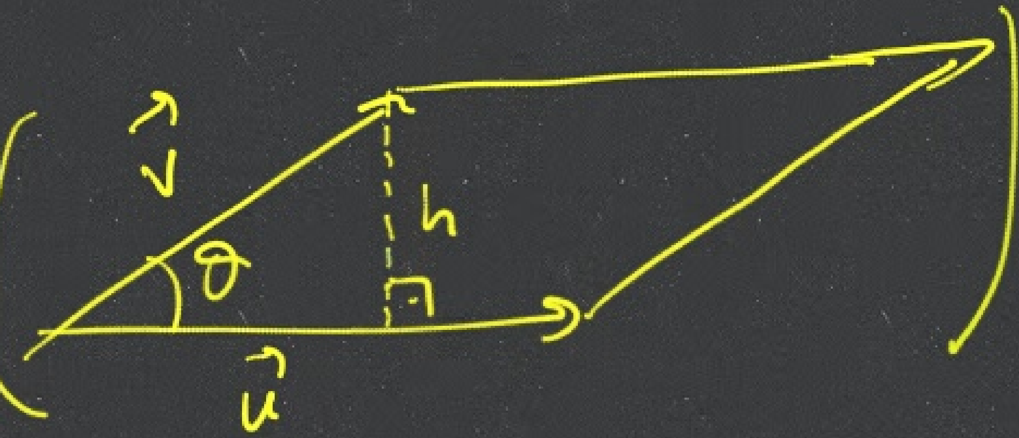
(6) $\vec u\cdot(\vec v\times\vec w) = \vec v\cdot(\vec w\times\vec u) = \vec w\cdot(\vec u\times\vec v)$

$\vec u, \vec v, \vec w, \vec u, \vec v$ order is important!

$\vec u\cdot(\vec w\times\vec v) = -\vec u\cdot(\vec v\times\vec w)$

AREA OF A PARALLELOGRAM in $\mathbb{R}^n$, $\underline{n \geq 2}$.

$\vec{u}, \vec{v}$ in $\mathbb{R}^n$;

$A = \text{Area}\Big($  $\Big) = |\vec{u}| \cdot h = |\vec{u}|\,|\vec{v}|\,\sin(\theta)$ $\left(\begin{array}{l}\text{when } n=3\\ = |\vec{u} \times \vec{v}|\end{array}\right)$

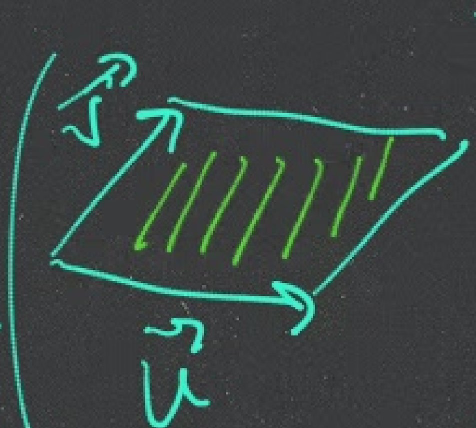
needs θ

can we compute it by dot product?

Recall that $\left\{\begin{array}{l} \vec{u} \cdot \vec{v} = |\vec{u}|\,|\vec{v}| \cdot \cos(\theta) \\ \vec{u} \cdot \vec{u} = |\vec{u}|^2 \quad \text{in } \mathbb{R}^n \end{array}\right\}$ (*) any n

$$A = |\vec{u}|\,|\vec{v}|\,\underbrace{\sqrt{1-\cos^2(\theta)}}_{\sin(\theta)} = \sqrt{|\vec{u}|^2|\vec{v}|^2 \cdot (1-\cos^2(\theta))} = \sqrt{|\vec{u}|^2|\vec{v}|^2 - |\vec{u}|^2|\vec{v}|^2\cos^2(\theta)}$$

$$\overset{(*)}{=} \sqrt{(\vec{u}\cdot\vec{u})(\vec{v}\cdot\vec{v}) - (\vec{u}\cdot\vec{v})^2}$$

$A = \text{Area}\Big($  $\Big) = \sqrt{(\vec{u}\cdot\vec{u})\,(\vec{v}\cdot\vec{v}) - (\vec{u}\cdot\vec{v})^2}$ for $\vec{u}, \vec{v} \in \mathbb{R}^n$

__Example:__ Find the area A of the parallelogram formed by the vectors $\vec{u} = 2\vec{e}_1 + 3\vec{e}_2 - \vec{e}_3 + \vec{e}_4$ & $\vec{v} = -\vec{e}_2 - 3\vec{e}_4$. $\left(A = \sqrt{(\vec{u}\cdot\vec{u})(\vec{v}\cdot\vec{v}) - (\vec{u}\cdot\vec{v})^2}\right.$.

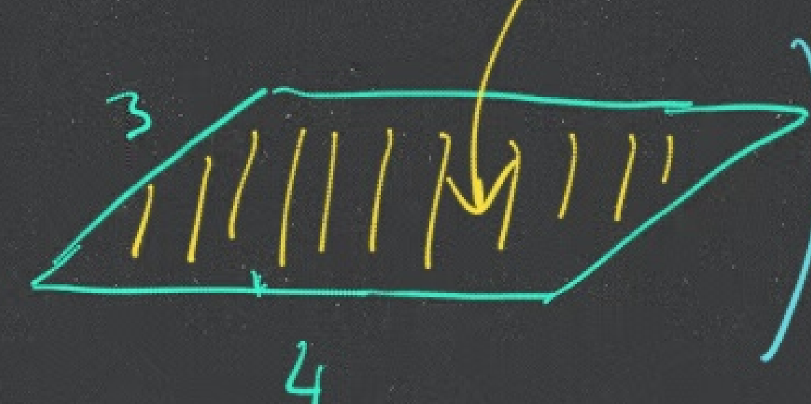
$n = 4$, $\underline{\vec{u}, \vec{v} \in \mathbb{R}^4}$

Have $\vec{u}\cdot\vec{u} = 2^2 + 3^2 + (-1)^2 + 1^2 = 15 = |\vec{u}|^2 \quad \Rightarrow |\vec{u}| = \sqrt{15} \approx 4$

$\vec{v}\cdot\vec{v} = (-1)^2 + (-3)^2 = 10 = |\vec{v}|^2 \quad \Rightarrow |\vec{v}| = \sqrt{10} \approx 3$

$\vec{u}\cdot\vec{v} = \cancel{2 \cdot 0} + 3\cdot(-1) + \cancel{(-1)\cdot 0} + 1\cdot(-3) = -6$

$$\Rightarrow A = \sqrt{15 \cdot 10 - 6^2} = \sqrt{150 - 36} = \sqrt{114} \approx 11$$

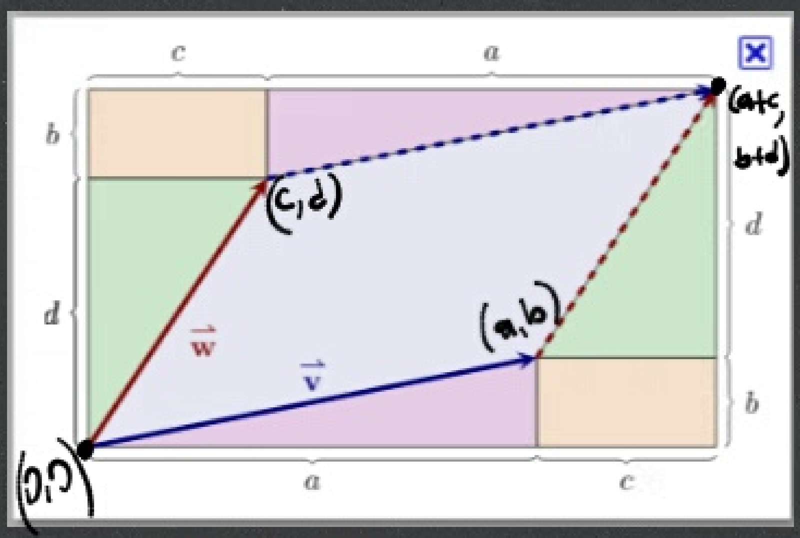
$A = \Big($  $\Big) = \sqrt{114} \leq |\vec{u}|\,|\vec{v}| = \sqrt{150}$

Determinants:

2×2 determinant

- $n=2$, $\begin{vmatrix} a & b \\ c & d \end{vmatrix} := ad-bc$ What is the significance of this number?

let $\vec{v} = a\vec{i} + b\vec{j}$, $\vec{w} = c\vec{i} + d\vec{j}$ be in $\mathbb{R}^2$.



xy-plane

A = Area of (parallelogram with sides $\vec{w}$, $\vec{v}$) = ?

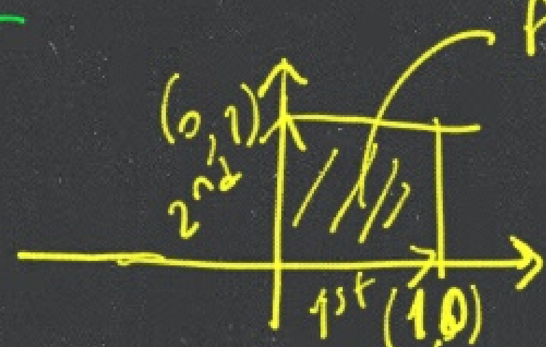
$\| \; |(a+c)(b+d) - [2bc + \tfrac{2}{2}ab + \tfrac{2}{2}cd]| =$

$= |\cancel{ab} + ad + bc + \cancel{cd} - 2bc - \cancel{ab} - \cancel{cd}| = |ad-bc| = \left| \begin{vmatrix} a & b \\ c & d \end{vmatrix} \right|$ ← $\vec{v}$, ← $\vec{w}$

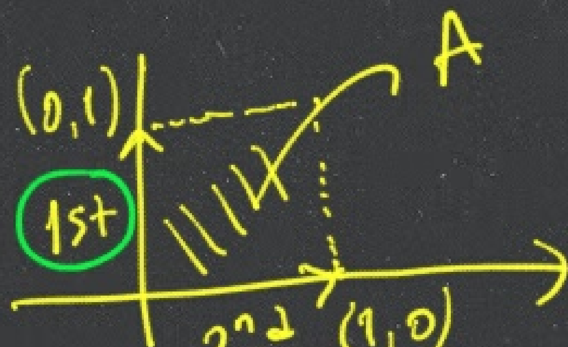
abs. value sign

* If area is zero, no parallelogram, then $\vec{u} \parallel \vec{v}$.

Conversely, if $\vec{u} \parallel \vec{v} \implies \begin{vmatrix} a & b \\ ta & tb \end{vmatrix} = abt - abt = 0$. So, $\boxed{\vec{u} \parallel \vec{v} \iff \begin{vmatrix} a & b \\ c & d \end{vmatrix} = 0}$



$\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 = A$, but



$\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1$

$A = |-1| = 1$

- $n=3$, 3×3 determinants.

$$D = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & \ell \end{vmatrix} = a(e\ell - fh) - b(d\ell - fg) + c(dh - eg)$$

$$= a\begin{vmatrix} e & f \\ h & \ell \end{vmatrix} - b\begin{vmatrix} d & f \\ g & \ell \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}.$$

Let V be the volume of the parallelpiped formed by $\vec{u}, \vec{v}, \vec{w}$, in $\mathbb{R}^3$ (take the parallelogram with sides $\vec{v}, \vec{w}$ as the base.)

$$\begin{cases} \vec{u} = a\vec{i} + b\vec{j} + c\vec{k} \\ \vec{v} = d\vec{i} + e\vec{j} + f\vec{k} \\ \vec{w} = g\vec{i} + h\vec{j} + \ell\vec{k} \end{cases}$$

$\vec{v} \times \vec{w} = \begin{vmatrix} e & f \\ h & \ell \end{vmatrix}\vec{i} - \begin{vmatrix} d & f \\ g & \ell \end{vmatrix}\vec{j} + \begin{vmatrix} d & e \\ g & h \end{vmatrix}k$

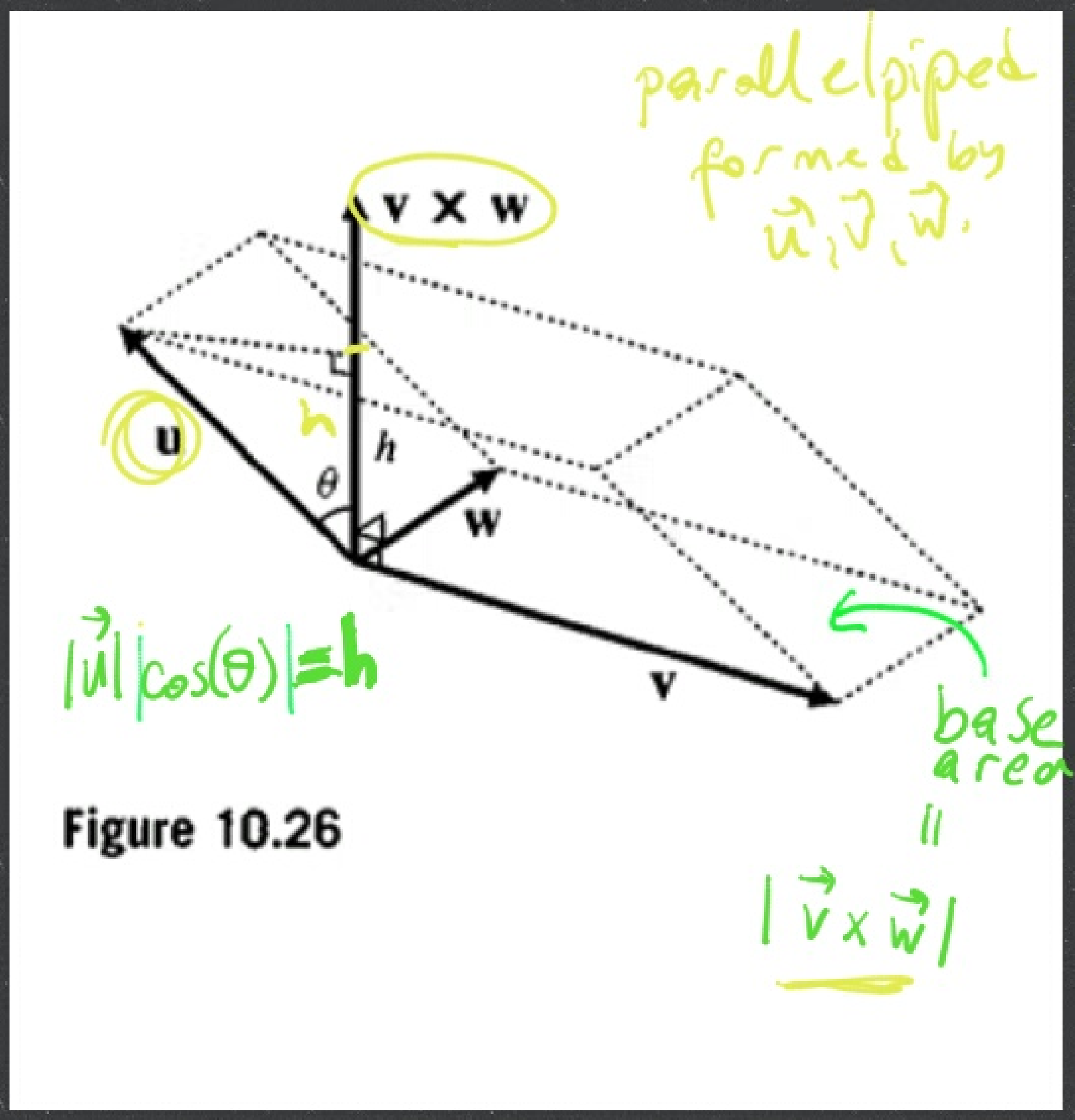


Figure 10.26

$0 \le V = (\text{base area}) \cdot \underset{h}{\text{height}}$.

$= |\, |\vec{v} \times \vec{w}| \cdot |\vec{u}| \cdot \cos(\theta) \,| = |\, \underbrace{(\vec{u} \cdot (\vec{v} \times \vec{w}))}_{\text{may be} \le 0} \,|$ ← absolute value

$\vec{u} \cdot (\vec{v} \times \vec{w}) = a\begin{vmatrix} e & f \\ h & \ell \end{vmatrix} + b\left(-\begin{vmatrix} d & f \\ g & \ell \end{vmatrix}\right) + c\begin{vmatrix} d & e \\ g & h \end{vmatrix} = D$

$\Rightarrow \boxed{V = \text{abs. value of } D = |D|}$

Application: CROSS PRODUCT AS A DETERMINANT

Note that if we write $\vec{i}, \vec{j}, \vec{k}$ symbolically for a, b, c

$$\underbrace{\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}}_{\vec{u}\times\vec{v}} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}\vec{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}\vec{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}\vec{k} = \vec{u}\times\vec{v}$$

where $\boxed{\vec{u} = u_1\vec{i} + u_2\vec{j} + u_3\vec{k}}$
$\boxed{\vec{v} = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}}$

So, $\vec{u}\times\vec{v}$ is easy to compute as a 3×3 - determinant.

$\boxed{4\times 4 \text{ determinant.}}$

- $n=4$,

$$\begin{vmatrix} \cancel{a} & \cancel{b} & \cancel{c} & \cancel{d} \\ e & f & g & h \\ p & q & r & s \\ t & x & y & z \end{vmatrix} := a\begin{vmatrix} f & g & h \\ q & r & s \\ x & y & z \end{vmatrix} - b\begin{vmatrix} e & g & h \\ p & r & s \\ t & y & z \end{vmatrix} + c\begin{vmatrix} e & f & h \\ p & q & s \\ t & x & z \end{vmatrix} - d\begin{vmatrix} e & f & g \\ p & q & r \\ t & x & y \end{vmatrix}$$

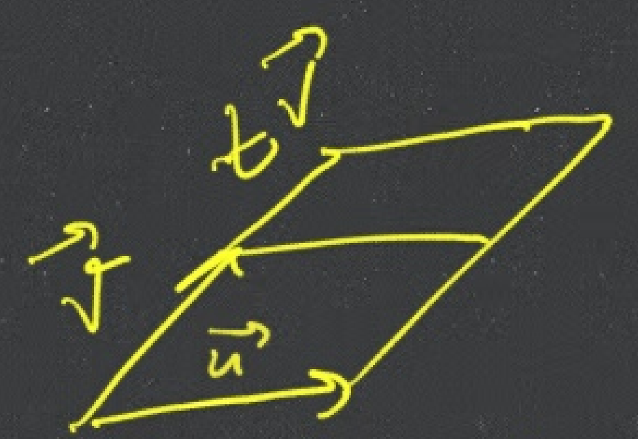
similarly for $n = 5, 6, 7 \ldots, \; - - -$

PROPERTIES OF DETERMINANTS
$n\times n$

We will write them for $n=2$, but true for any $n \geq 2$. $\left\{\begin{array}{l} \vec{u} = a\vec{i} + b\vec{j} \\ \vec{v} = c\vec{i} + d\vec{j} \end{array}\right\}$

① interchanging two rows (or two columns) multiplies the determinant by -1.

$$\underset{\displaystyle -\begin{vmatrix} b & a \\ d & c \end{vmatrix}}{\underset{\shortparallel}{ad-bc}} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = -\begin{vmatrix} c & d \\ a & b \end{vmatrix} = -(bc-ad)$$



② Multiplying a row (or column) by a scalar t, multiplies the determinant by t.

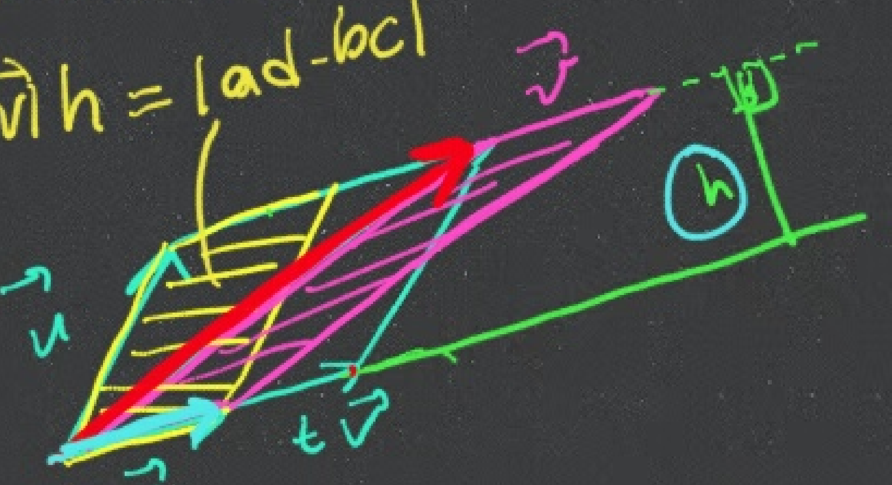
$$\underset{t(ad-bc)}{atd-bct} = \begin{vmatrix} a & b \\ tc & td \end{vmatrix} = t\cdot\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & tb \\ c & td \end{vmatrix} = atd - ctb = t(ad-bc).$$

③ $$\begin{vmatrix} a & b \\ c & d \end{vmatrix} \overset{(*)}{=} \begin{vmatrix} a+tc & b+dt \\ c & d \end{vmatrix} = ad + \cancel{dtc} - bc - \cancel{dtc} \left(= \begin{vmatrix} \vec{u}+t\vec{v} \\ \vec{v} \end{vmatrix} \right)$$
$= ad - bc$

\\ yellow area

\\ pink area

$|\vec{v}|h = |ad-bc|$



red vector $\vec{u}+t\vec{v}$

④ $$\begin{vmatrix} a & b \\ ta & tb \end{vmatrix} \overset{(2)}{=} t\begin{vmatrix} a & b \\ a & b \end{vmatrix} = 0$$

no area $\xrightarrow{\vec{u}} \xrightarrow{t\vec{u}}$.

(*) a scalar multiple of one row added to another row does not change the determinant.

Expansion of determinant using any row or column gives the same result as long as the signs are applied properly.

e.g. $\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$ meaning that

2nd Column

2nd row

3rd row

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = -d\begin{vmatrix} b & c \\ h & i \end{vmatrix} + e\begin{vmatrix} a & c \\ g & i \end{vmatrix} - f\begin{vmatrix} a & b \\ g & h \end{vmatrix} = g\begin{vmatrix} b & c \\ e & f \end{vmatrix} - h\begin{vmatrix} a & c \\ d & f \end{vmatrix} + i\begin{vmatrix} a & b \\ d & e \end{vmatrix}$$

expansion w.r.to 2nd row

expansion w.r.to 3-rd row

$$= -dbi + dhc + aei - egc - afh + fgb$$

$$\| \quad -b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + e\begin{vmatrix} a & c \\ g & i \end{vmatrix} - h\begin{vmatrix} a & c \\ d & f \end{vmatrix} = -bdi + bgf + aei - egc - haf + hdc$$

expansion w.r.to 2nd column.

Examples

(1) $$\begin{vmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 7 \end{vmatrix} = 1 \cdot \underbrace{\begin{vmatrix} 4 & 5 \\ 0 & 7 \end{vmatrix}}_{=28} + 0 + 0 = 28.$$

(2) $$\begin{vmatrix} 0 & 0 & 7 \\ 0 & 4 & 5 \\ 1 & 2 & 3 \end{vmatrix} = +1 \cdot \begin{vmatrix} 0 & 7 \\ 4 & 5 \end{vmatrix} = -28$$

(3) $$\begin{vmatrix} 1 & 2 & 3 \\ 0 & 12 & 15 \\ 0 & 0 & 14 \end{vmatrix} = 3\begin{vmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 14 \end{vmatrix} = (3 \cdot 2)\begin{vmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 7 \end{vmatrix} = 6 \cdot 28 = 168$$

$$\| \quad +14\begin{vmatrix} 1 & 2 \\ 0 & 12 \end{vmatrix} = 14 \cdot 12 = 168.$$

(4) $$28 = \begin{vmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 7 \end{vmatrix} = 3\underbrace{\begin{vmatrix} 0 & 4 \\ 0 & 0 \end{vmatrix}}_{=0} - 5\underbrace{\begin{vmatrix} 1 & 2 \\ 0 & 0 \end{vmatrix}}_{=0} + 7\begin{vmatrix} 1 & 2 \\ 0 & 4 \end{vmatrix} = 7 \cdot 4 = 28$$

3rd column

* Volume of the parallelpiped formed by the vectors $\vec{u}, \vec{v}, \vec{w}$ in $\mathbb{R}^3$ is the absolute value of the 3×3 determinant

$$\text{Determ}\begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = u_1\begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} + u_2\left(-\begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix}\right) + u_3\begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix} = \vec{u}\cdot\underbrace{(\vec{v}\times\vec{w})}_{}$$

triple scalar product

$\therefore$ $\vec{u}, \vec{v}, \vec{w}$ lie on the same plane $\Leftrightarrow$ volume of the parallelpiped is 0.

* $\Leftrightarrow \vec{u}, \vec{v}, \vec{w}$ are coplanar.

$\Leftrightarrow$ $\vec{u}\cdot(\vec{v}\times\vec{w})$ scalar triple product of $\vec{u}, \vec{v}, \vec{w}$.

As expected $\vec{u}\cdot(\vec{v}\times\vec{w}) = \vec{v}\cdot(\vec{w}\times\vec{u}) = \vec{w}\cdot(\vec{u}\times\vec{v})$.

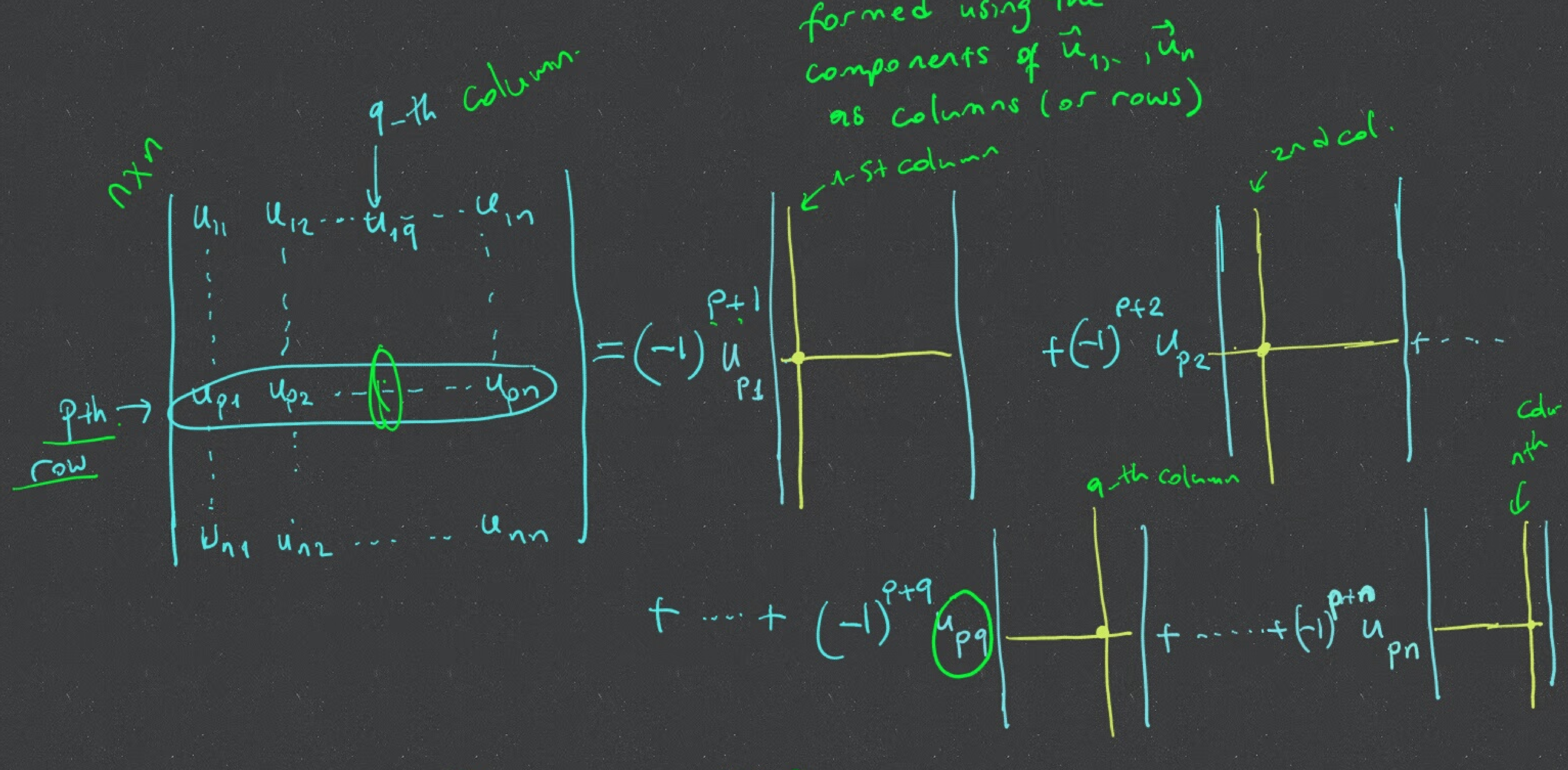
$\vec{u}\cdot(\vec{v}\times\vec{w}) = -\vec{u}\cdot(\vec{w}\times\vec{v})$

VOLUME OF AN n-PARALLELOTOPE.

* Let $\vec{u}_1, \vec{u}_2, \dots, \vec{u}_n$ be vectors in $\mathbb{R}^n$, the volume of the n-parallelotope formed by $\vec{u}_1, \dots, \vec{u}_n$ is the absolute value of the $n\times n$ determinant whose rows are components of $\vec{u}_i$'s.

if $\vec{u}_i = u_{i1}\vec{e}_1 + \dots + u_{in}\vec{e}_n \Rightarrow \boxed{\text{Volume}} = \begin{vmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ u_{21} & u_{22} & \cdots & u_{2n} \\ \vdots & \vdots & & \vdots \\ u_{n1} & u_{n2} & \cdots & u_{nn} \end{vmatrix}$.

$\|$ absolute value of the $n\times n$ determinant formed using the components of $\vec{u}_1, \dots, \vec{u}_n$ as columns (or rows)



End of 10.3