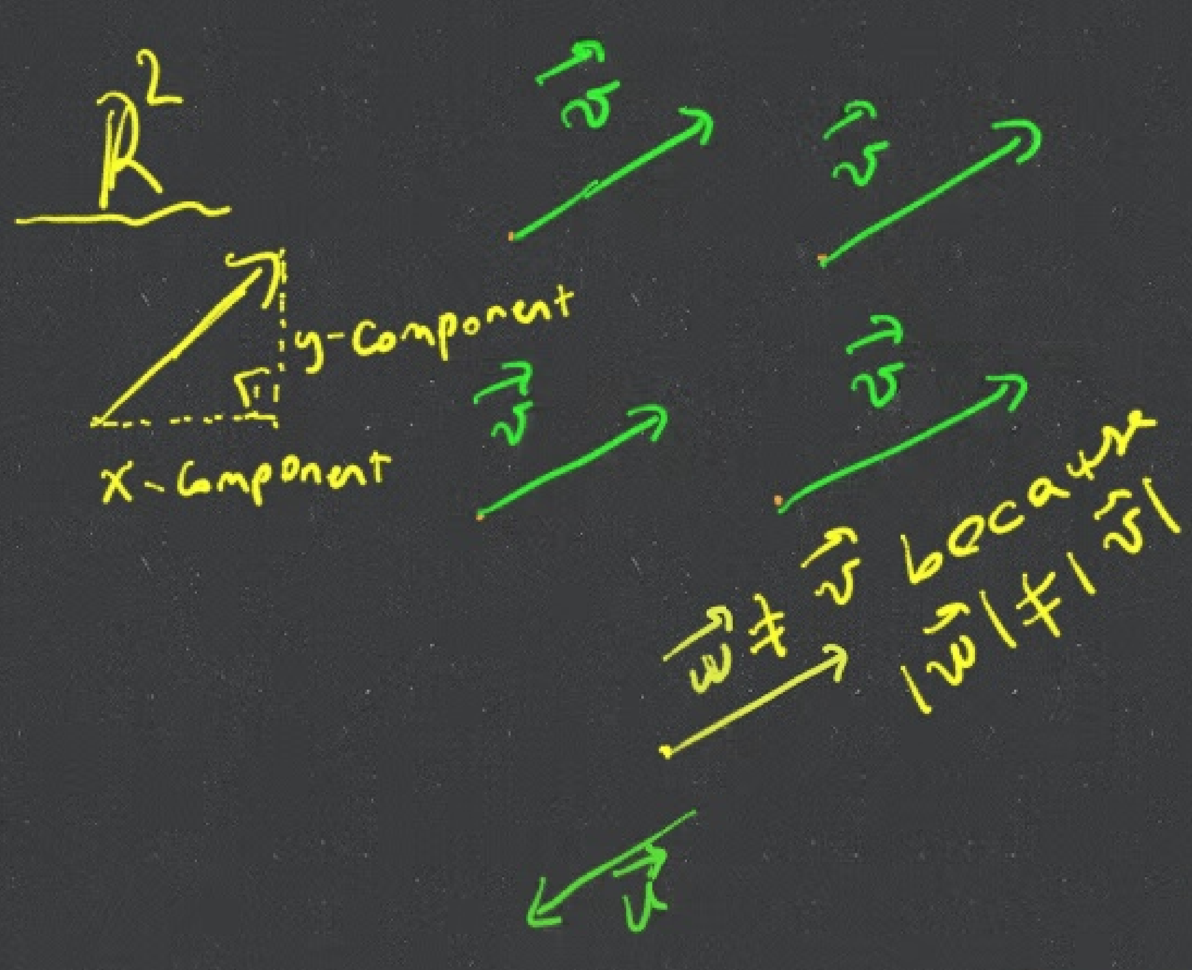
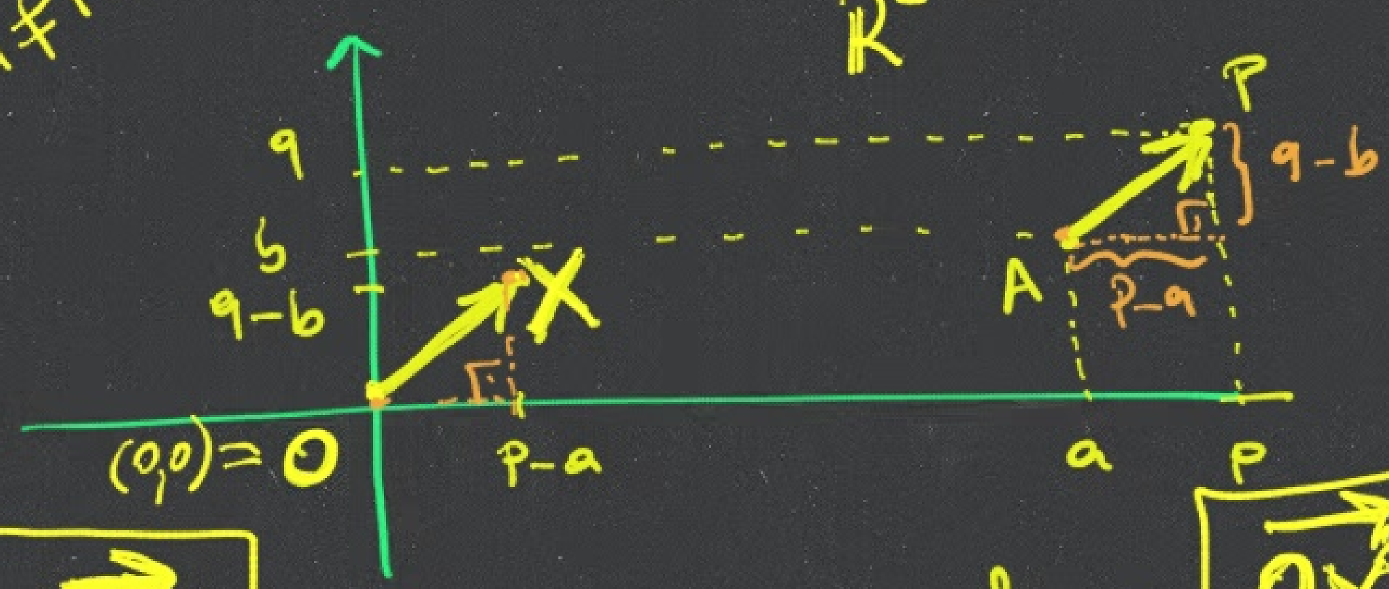
10.2. Vectors

A **vector** $\vec{v}$ is a quantity that involves a magnitude (size or length) denoted by $|\vec{v}|$ (or $\|\vec{v}\|$) and a direction. It does not have a position.



Two vectors $\vec{v}$, $\vec{w}$ are considered equal if they have the same length and the same direction.



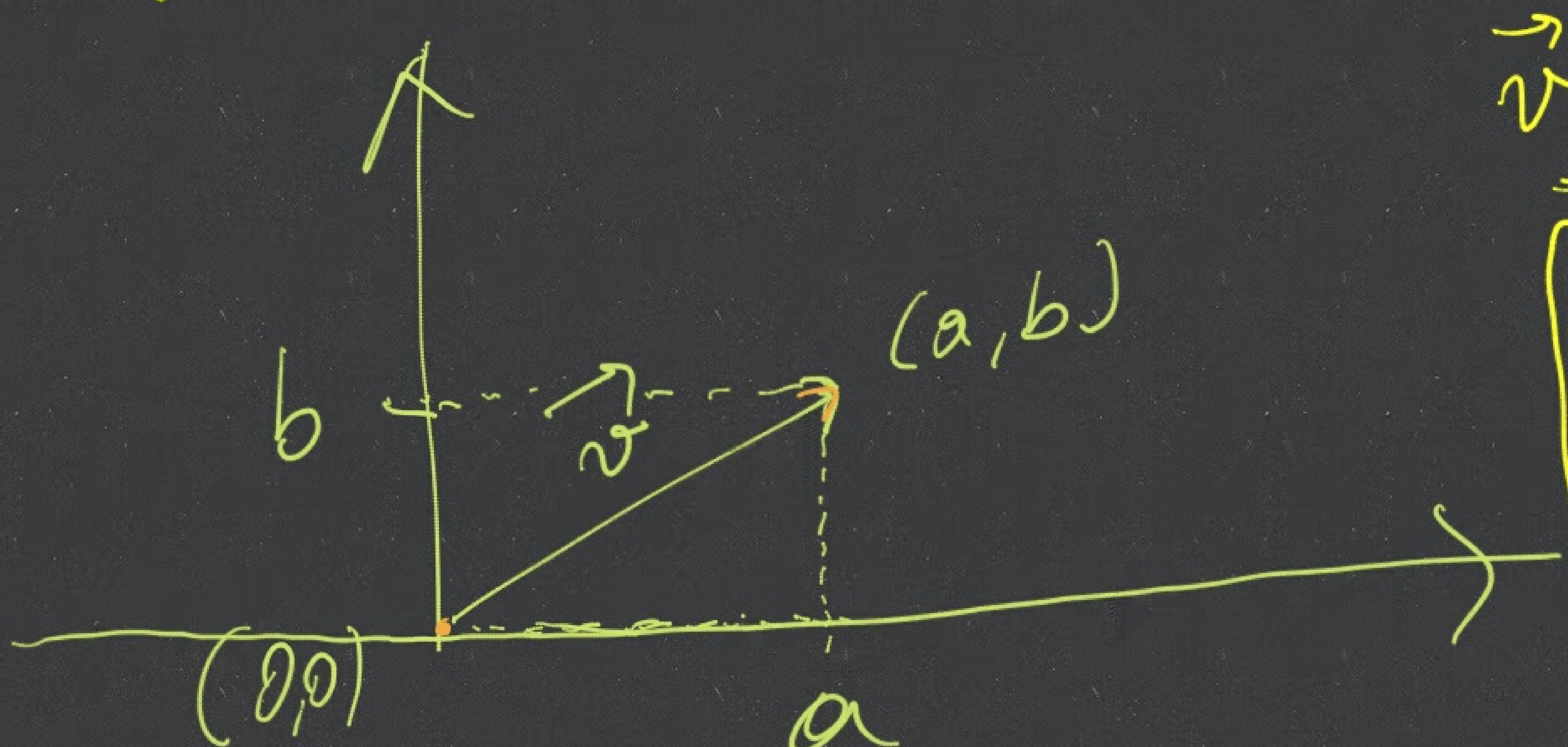
$A=(a,b)$

$P=(p,q)$

$X=(p-a,\ q-b)$

The vector $\boxed{\overrightarrow{AP}}$ is the equal to $\boxed{\overrightarrow{OX}}$ because they have the same direction and the same length which is

$$\sqrt{(p-a)^2+(q-b)^2} = |\overrightarrow{AP}| = |\overrightarrow{OX}|$$

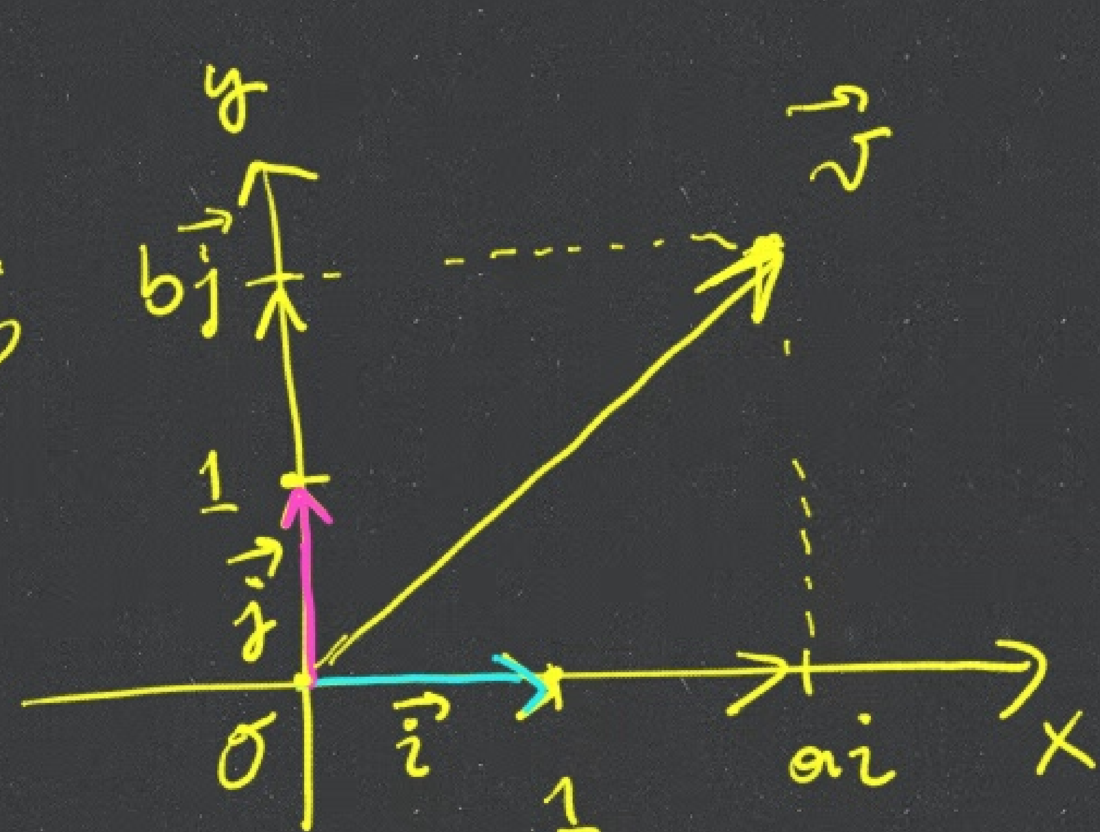


$\vec{v} = \overrightarrow{O(a,b)}$

$$\boxed{|\vec{v}| = \sqrt{a^2+b^2}}$$

To simplify notation $\vec{v} = \overrightarrow{O(a,b)}$;

set $\vec{i} = \overrightarrow{O(1,0)}$, $\vec{j} = \overrightarrow{O(0,1)}$



$$\vec{v} = a\vec{i} + b\vec{j}$$

⊛ Special case: $(a,b)=(0,0)$ is called as the zero vector even though it has no direction and has length zero, denoted by $\vec{0}$.

Vector addition & scalar multiplication:

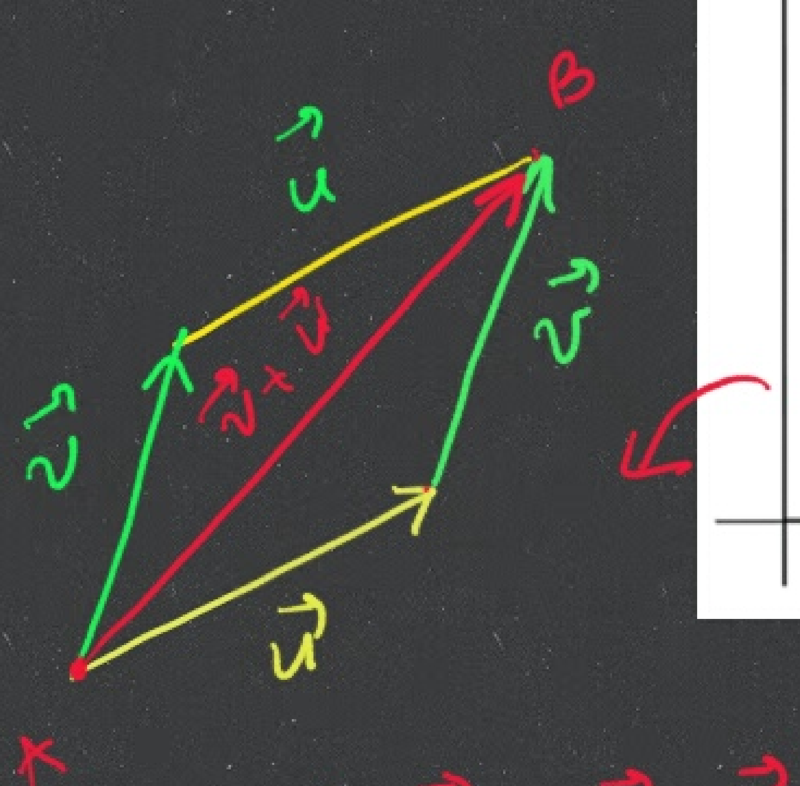
· Let $\vec{u}=a\vec{i}+b\vec{j}$, $\vec{v}=x\vec{i}+y\vec{j}$; then $\vec{u}+\vec{v}=(a+x)\vec{i}+(b+y)\vec{j}$,

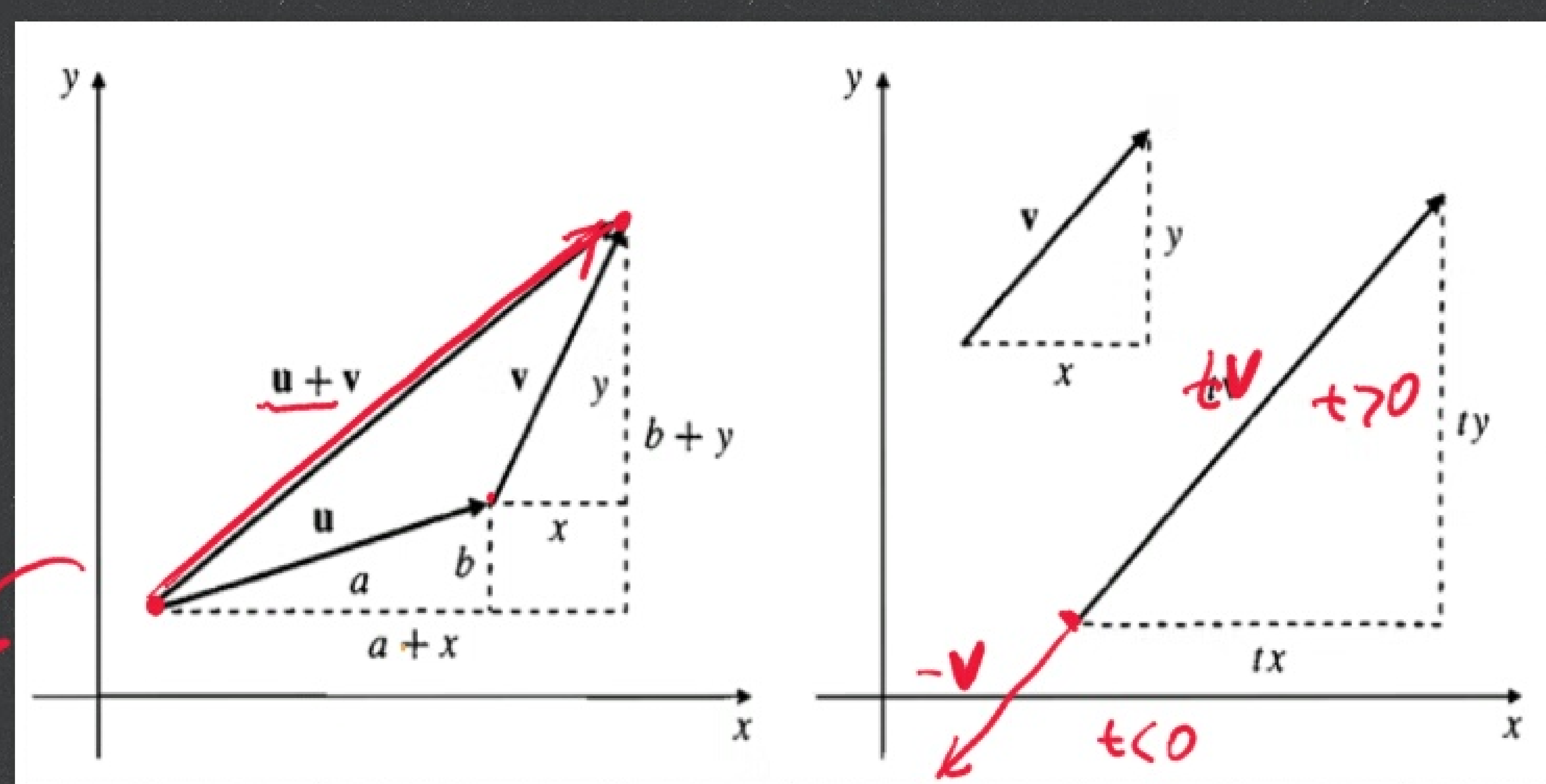
and $\vec{v}-\vec{u}=(x-a)\vec{i}+(y-b)\vec{j}$; $\left(\vec{v}-\vec{v}=\vec{0}=0\cdot\vec{i}+0\cdot\vec{j}\right)$

for scalar $t\in\mathbb{R}$, $t\cdot\vec{v}=tx\vec{i}+ty\vec{j}$; $\left(0\cdot\vec{v}=\vec{0}\right)$ scalar multiplic.

$|t\cdot\vec{v}|=|t||\vec{v}|$

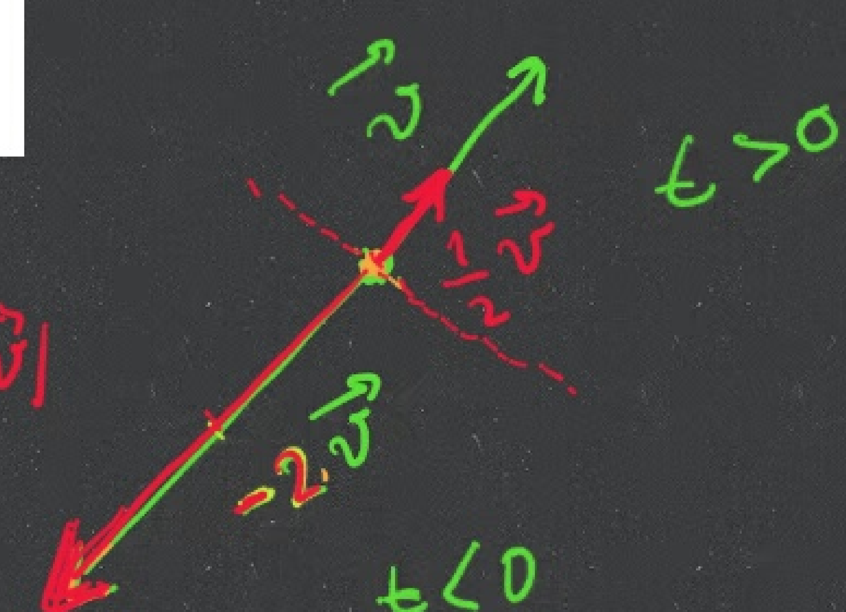
$\boxed{\text{Then } \vec{v}-\vec{u}=\vec{v}+(-\vec{u}).}$

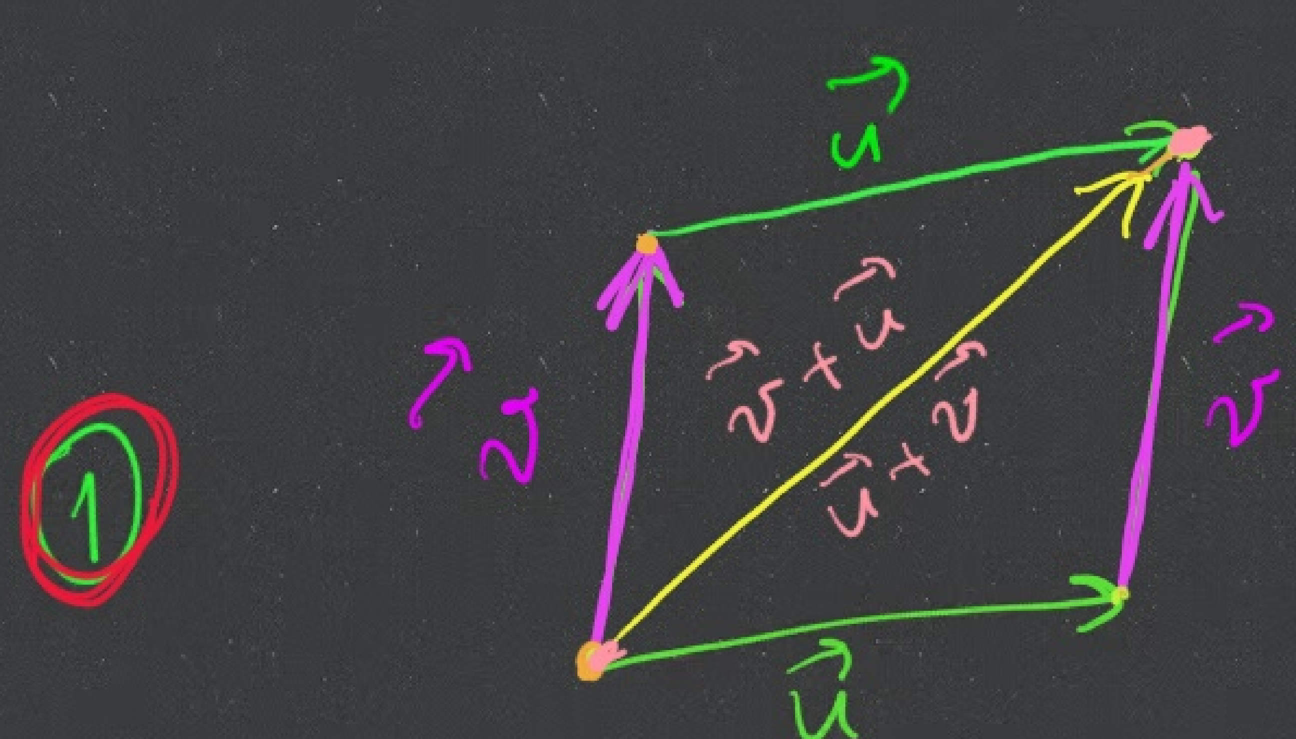


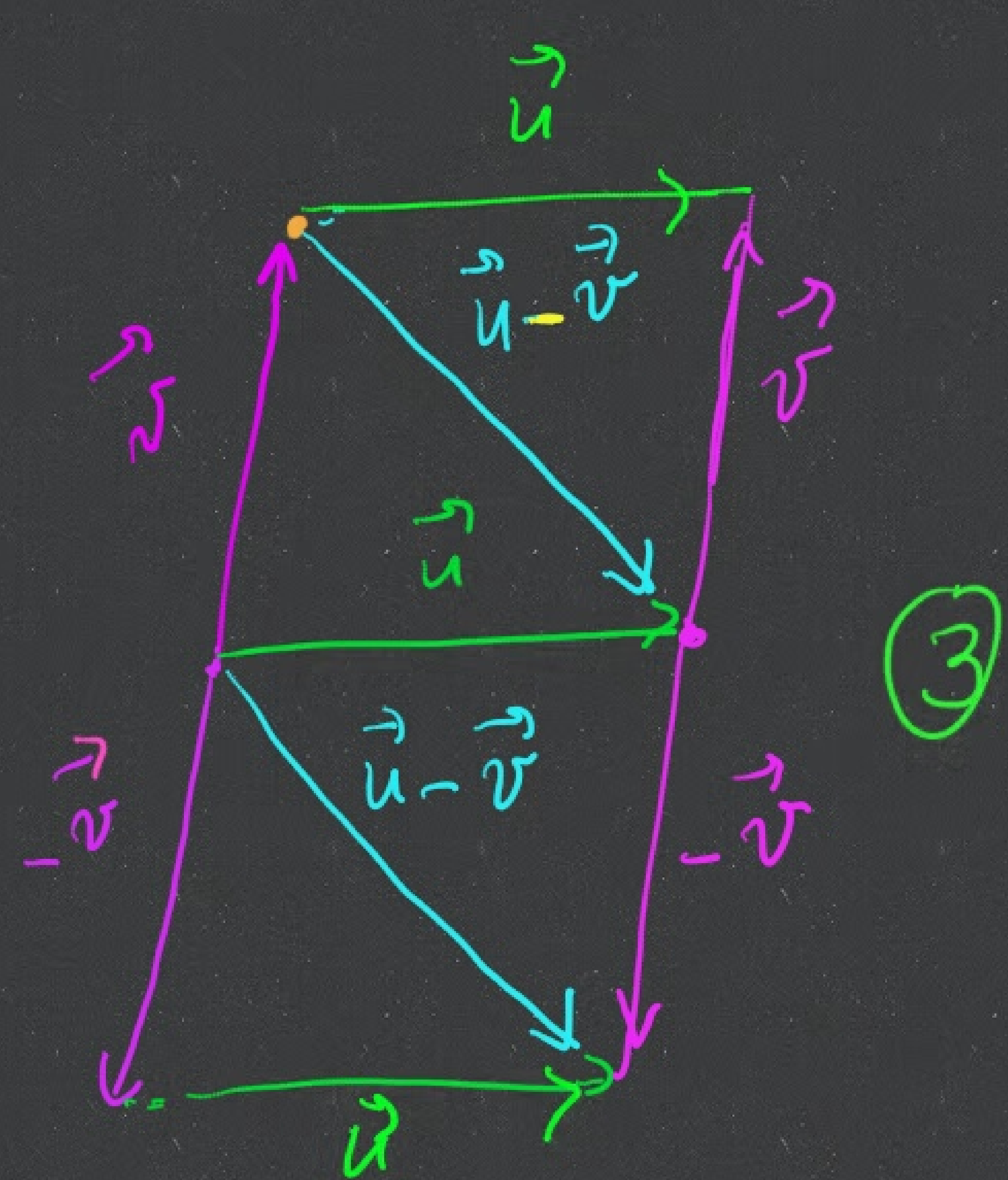


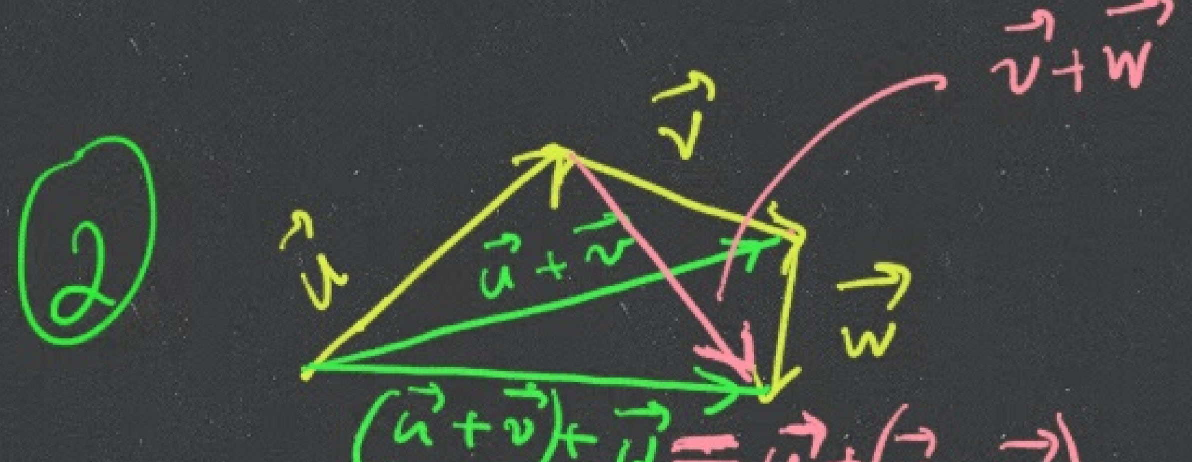
$\vec{u}+\vec{v}=\vec{v}+\vec{u}$

$2|\vec{v}|=|-2\vec{v}|$









$(\vec{u}+\vec{v})+\vec{w}=\vec{u}+(\vec{v}+\vec{w})$

We can summarize these as follows.

commutativity — ① $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$,

associativity — ② $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$,

③ $\mathbf{u}-\mathbf{v}=\mathbf{u}+(-1)\mathbf{v}$,

④ $t(\mathbf{u}+\mathbf{v})=t\mathbf{u}+t\mathbf{v}$.

④ $t(\vec{u}+\vec{v})=$

$=t[(a+x)\vec{i}+t(b+y)\vec{j}]$

$=(ta+tx)\vec{i}+(tb+ty)\vec{j}$

$=ta\vec{i}+tx\vec{i}+tb\vec{j}+ty\vec{j}$

$=ta\vec{i}+tb\vec{j}+tx\vec{i}+ty\vec{j}$

$=t\cdot\vec{u}+t\vec{v}.$

A **linear combination** of vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ in $\mathbb{R}^n$

a vector, $t_1 \vec{v}_1 + \cdots + t_k \vec{v}_k$ where $t_1, \ldots, t_k \in \mathbb{R}$.

Example: $\vec{v} = x \cdot \vec{i} + y \vec{j}$ is a linear combination of $\vec{i}$ & $\vec{j}$; x, y are in $\mathbb{R}$ scalars.

Example: $\vec{v} \in \mathbb{R}^3$; let $\vec{k} = \overrightarrow{O(0,0,1)}$, then

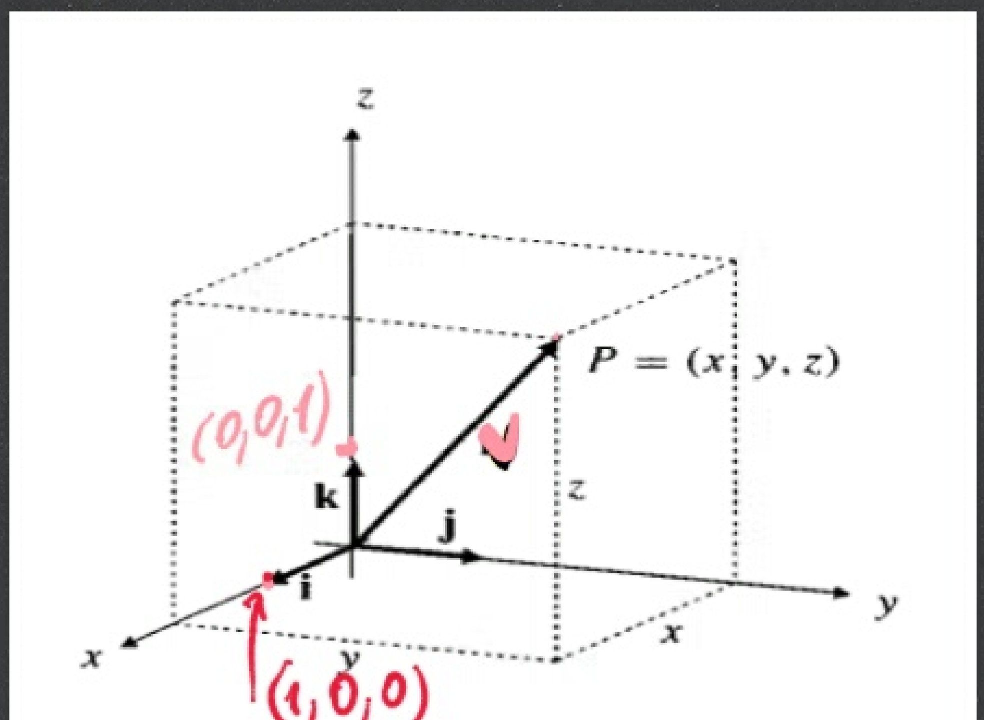


Figure 10.17 The standard basis vectors **i**, **j**, and **k**

$\vec{v} = \overrightarrow{OP} = x\vec{i} + y\vec{j} + z\vec{k}$.

$\vec{v}$ is a linear combination of $\vec{i}, \vec{j}, \vec{k}$.

$\vec{i} = \overrightarrow{O(1,0,0)} \in \mathbb{R}^3$

$\vec{j} = \overrightarrow{O(0,1,0)} \in \mathbb{R}^3$

---

a **Unit vector** $\equiv$ a vector of length 1. Examples: $|\vec{i}| = 1$, $|\vec{j}| = 1$, $|\vec{k}| = 1$

- If $\vec{v}$ is any vector with length $|\vec{v}|$, then

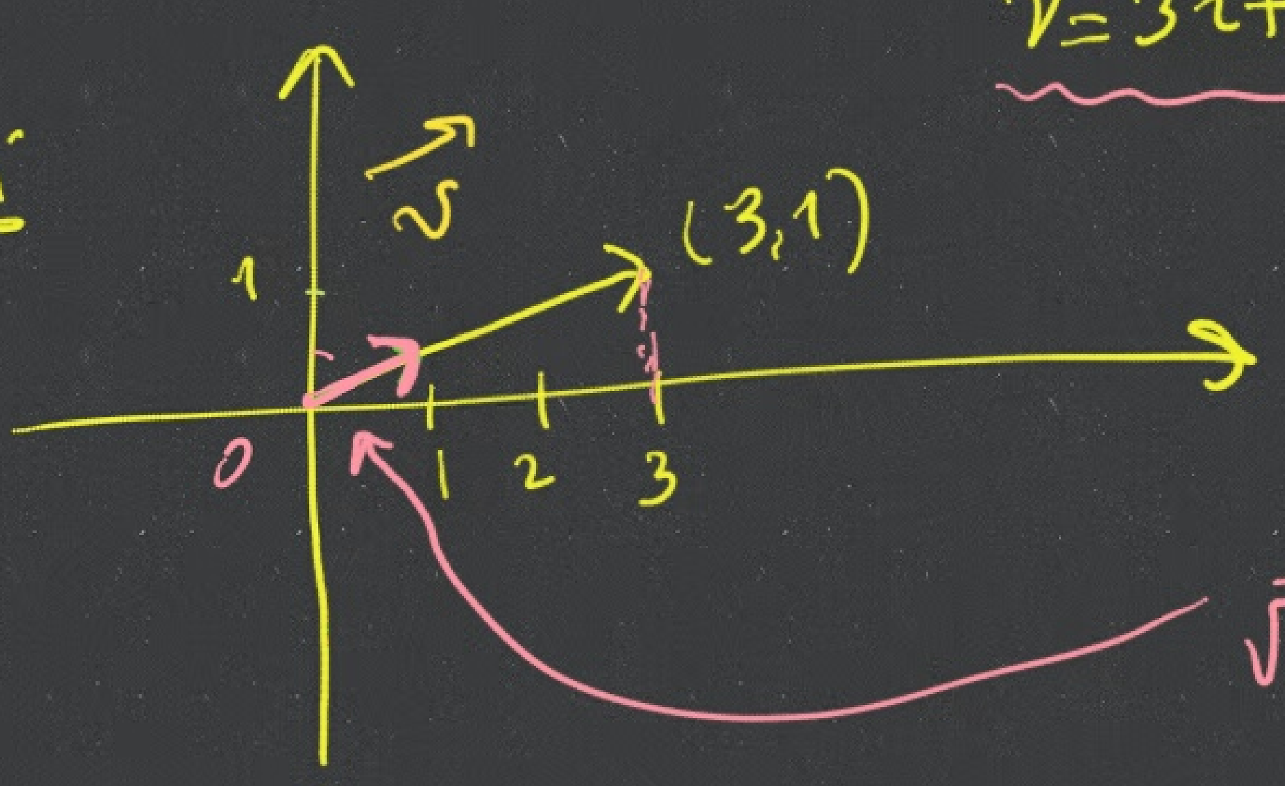
$$|t \vec{v}| \overset{*}{=} |t| \, |\vec{v}| \quad \text{for any } t \in \mathbb{R}.$$

If $|\vec{v}| \neq 0$, then $\dfrac{1}{|\vec{v}|} \cdot \vec{v}$ is a unit vector

because $\left| \left(\dfrac{1}{|\vec{v}|}\right) \vec{v} \right| \overset{*}{=} \dfrac{1}{\cancel{|\vec{v}|}} \cdot \cancel{|\vec{v}|} = 1$

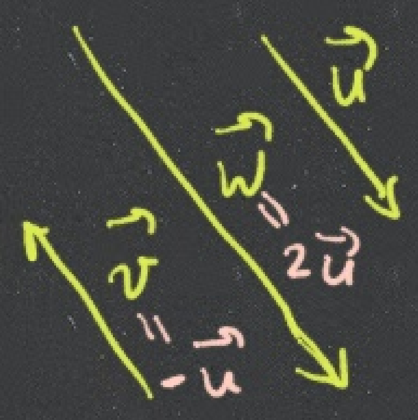
$\hat{\vec{v}}$ denotes the unit vector in the direction of $\vec{v}$

Example: $\vec{v} = 3\vec{i} + \vec{j} \Rightarrow |\vec{v}| = \sqrt{9+1} = \sqrt{10}$



$\Rightarrow \hat{\vec{v}} = \underbrace{\dfrac{3}{\sqrt{10}} \vec{i} + \dfrac{1}{\sqrt{10}} \vec{j}}_{\text{unit vector along } \vec{v}}$

Definition: Two vectors $\vec{u}$, $\vec{v}$ are called parallel if $\vec{u} = t\vec{v}$ for some $t \in \mathbb{R}$; if this is the case we write $\vec{u} \parallel \vec{v}$.



Examples: $\vec{u} = \vec{i} - 2\vec{j}$ & $\vec{v} = -\vec{i} + 2\vec{j}$ & $\vec{w} = 2\vec{i} - 4\vec{j}$

$\underbrace{\vec{v} = -\vec{u}}$, $\underbrace{\vec{w} = 2\cdot\vec{u}}$, $\underbrace{\vec{v} = -\frac{1}{2}\vec{w}}$ so that $\underbrace{\vec{u} \parallel \vec{v}}$, $\vec{v} \parallel \vec{w}$, $\underbrace{\vec{u} \parallel \vec{w}}$

- The DOT PRODUCT OF vectors:

$\vec{u} = u_1\vec{i} + u_2\vec{j} + u_3\vec{k}$, $\vec{v} = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$ in $\mathbb{R}^3$.

Then $\vec{u} \bullet \vec{v} := u_1v_1 + u_2v_2 + u_3v_3$ is a real number.

$\vec{u} \cdot \vec{u} = u_1^2 + u_2^2 + u_3^2 = |\vec{u}|^2$

(Dot $\equiv$ scalar $\equiv$ inner)

$|\vec{u}| := \sqrt{\vec{u} \cdot \vec{u}}$

$\mathbf{u} \bullet \mathbf{v} = \mathbf{v} \bullet \mathbf{u}$	①	(commutative law),
$\mathbf{u} \bullet (\mathbf{v} + \mathbf{w}) = \mathbf{u} \bullet \mathbf{v} + \mathbf{u} \bullet \mathbf{w}$	②	(distributive law),
$(t\mathbf{u}) \bullet \mathbf{v} = \mathbf{u} \bullet (t\mathbf{v}) = t(\mathbf{u} \bullet \mathbf{v})$	③	(for real t),
$\mathbf{u} \bullet \mathbf{u} = \lvert\mathbf{u}\rvert^2$.	④	

Let $\theta \in [0, \pi]$ be the angle between $\vec{u}$ and $\vec{v}$.

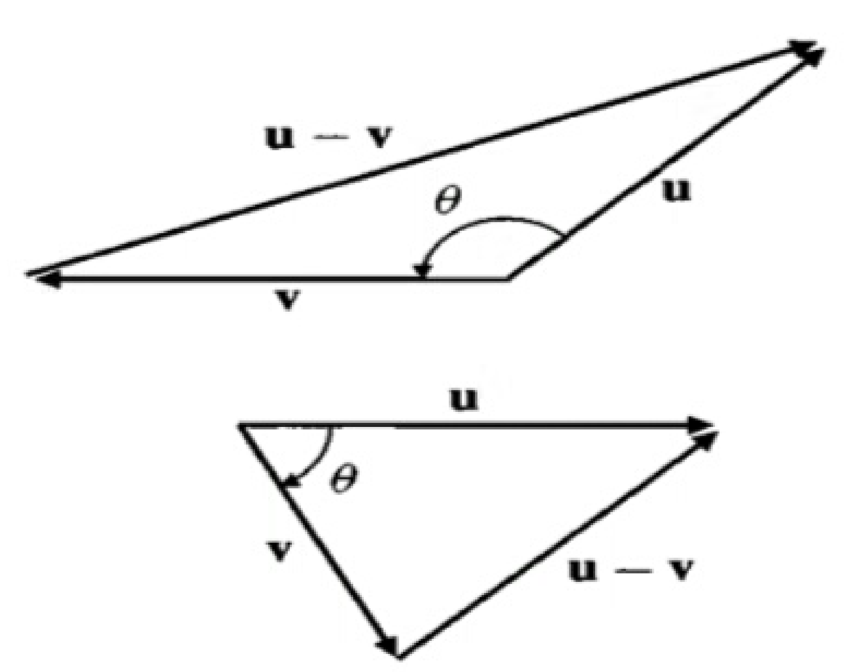


Figure 10.20 Applying the Cosine Law to a triangle reveals the relationship between dot the product and angle between vectors

Law of cosines gives

$|\vec{u}|^2 + |\vec{v}|^2 - 2|\vec{u}||\vec{v}| \cdot \cos\theta =$

$= |\vec{u} - \vec{v}|^2 = (\vec{u} - \vec{v}) \bullet (\vec{u} - \vec{v})$

$= \vec{u} \cdot (\vec{u} - \vec{v}) - \vec{v} \cdot (\vec{u} - \vec{v})$ (properties of $\cdot$)

$= |\vec{u}|^2 - \vec{u}\cdot\vec{v} - \vec{v}\cdot\vec{u} + |\vec{v}|^2$

$\Rightarrow \cancel{-2}|\vec{u}||\vec{v}|\cos\theta = \cancel{-2}\vec{u} \bullet \vec{v}$ //

$\Rightarrow \boxed{\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}| \cdot \cos(\theta)}$. Theorem 1

$\Downarrow$

$\theta = \arccos\left(\frac{\vec{u} \cdot \vec{v}}{|\vec{u}||\vec{v}|}\right)$ is the angle between $\vec{u}$ & $\vec{v}$.

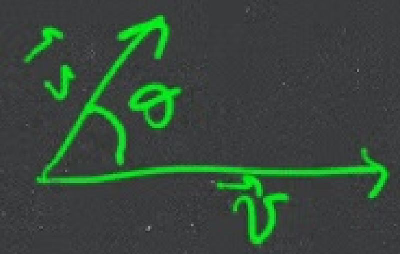
Example: If $\vec{u} = 2\vec{i} + \vec{j} - 2\vec{k}$ and $\vec{v} = 3\vec{i} - 2\vec{j} - \vec{k}$, find the angle between $\vec{u}$ & $\vec{v}$.

$\vec{u} \bullet \vec{v} = 2\cdot 3 + 1(-2) + (-2)(-1) = 6 = |\vec{u}||\vec{v}| \cdot \cos(\theta)$

$|\vec{u}| = \sqrt{4+1+4} = 3$, $|\vec{v}| = \sqrt{9+4+1} = \sqrt{14}$ //

$\Rightarrow \boxed{\cancel{6}^{\,2} = \cancel{3} \cdot \sqrt{14} \cdot \cos(\theta)} \Rightarrow \frac{2}{\sqrt{14}} = \cos(\theta) \approx$

$\Rightarrow \theta = \arccos\left(\frac{2}{\sqrt{14}}\right) (\approx 57.69^\circ)$



Two vectors $\vec u$ & $\vec v$ are perpendicular / orthogonal $\Leftrightarrow \boxed{\vec u \cdot \vec v = 0} \Leftrightarrow \boxed{\theta \text{ is } \pi/2}$

because, $\vec u \cdot \vec v = |\vec u||\vec v| \cdot \cos(\theta)$.

Projection of $\vec u$ on $\vec v$:

Vector projection of $\vec u$ on $\vec v$ is the vector $\boxed{\vec u_{\vec v}}$

Scalar projection of $\vec u$ on $\vec v$ is the number $s = |\vec u_{\vec v}|$.
$\qquad\qquad\qquad\qquad$ $\parallel$ length of $\vec u_{\vec v}$.

$\vec u_{\vec v} \parallel \vec v$

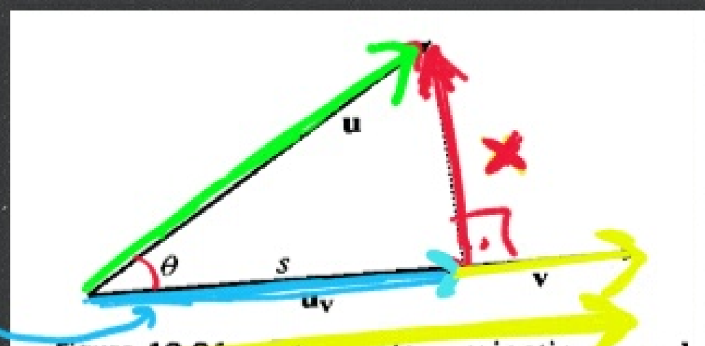
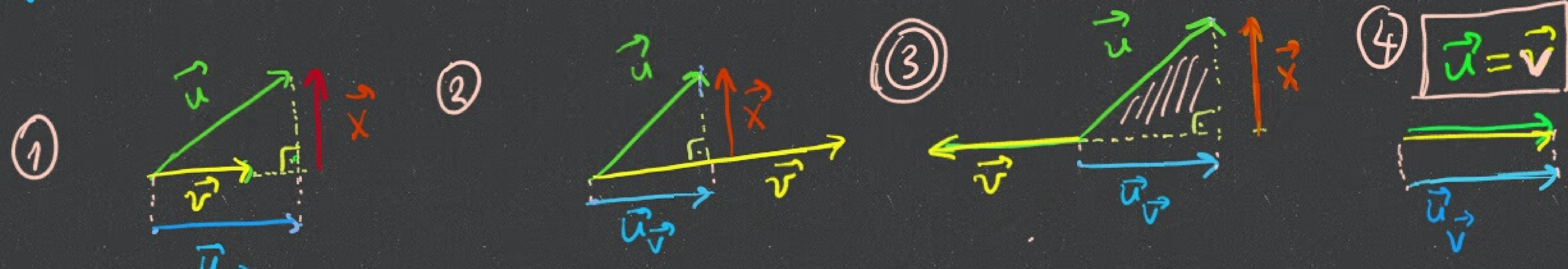


Figure 10.2… …e scalar projection s and the vector projection $\mathbf{u_v}$ of vector $\mathbf{u}$ along vector $\mathbf{v}$



④ $\boxed{\vec u = \vec v} \Rightarrow \vec u = \vec u_{\vec v}$, $\vec x = \vec 0$

$$\underbrace{\vec u_{\vec v} + \vec x}_{\text{perpendicular vectors.}} = \vec u \Rightarrow \boxed{\vec x = \vec u - \vec u_{\vec v}}$$

How to find a formula for the projection $\vec u_{\vec v}$ of $\vec u$ on $\vec v$?

$\vec u_{\vec v} \parallel \vec v$ and $s = |\vec u_{\vec v}| \Rightarrow \vec u_{\vec v} = s \cdot \hat{v} = \left(s \frac{1}{|\vec v|}\right) \cdot \vec v$ (unit vector along $\vec v$.)

$\cos(\theta) = \frac{s}{|\vec u|} \Rightarrow s = |\vec u| \cos(\theta)$

$$\Rightarrow \boxed{\vec u_{\vec v} = \frac{|\vec u|\cos(\theta)}{|\vec v|} \vec v}$$

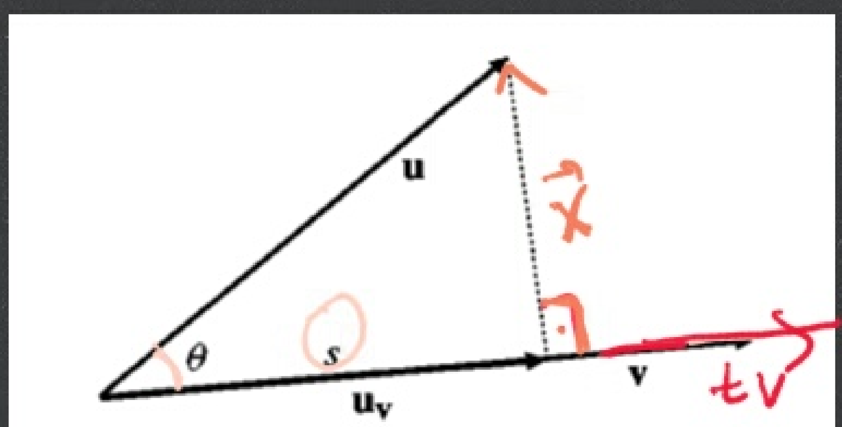


Figure 10.21 The scalar projection s and the vector projection $\mathbf{u_v}$ of vector $\mathbf{u}$ along vector $\mathbf{v}$

$$\vec u_{\vec v} = \left(\frac{|\vec u|\cos(\theta)}{|\vec v|} \frac{|\vec v|}{|\vec v|}\right)\vec v = \boxed{\frac{\vec u \cdot \vec v}{|\vec v|^2} \vec v = \vec u_{\vec v}}$$

$\boxed{\vec x = \vec u - \vec u_{\vec v}}$, $\vec u = \vec u_{\vec v} + \vec x$

Remark: $\vec u_{t\vec v} = \frac{\vec u \cdot (t\vec v)}{|t\vec v|^2} \cdot t\vec v = \frac{\cancel{t^2}\, \vec u \cdot \vec v}{\cancel{t^2}\, |\vec v|^2} \cdot \vec v = \vec u_{\vec v}$ for $t \in \mathbb{R} \setminus 0$.

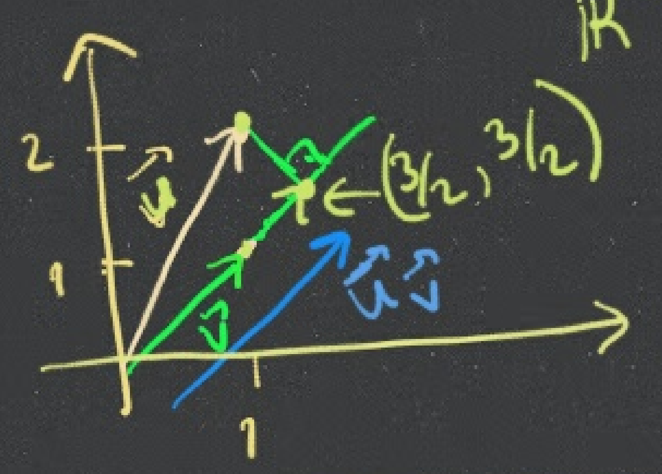
That is, the projection of $\vec u$ on $t\vec v$ for $t \in \mathbb{R} \setminus 0$

is the same as the " " " $\vec v$.

Example: Let $\vec{u} = \vec{i} + 2\vec{j}$, $\vec{v} = \vec{i} + \vec{j}$.

(1) Find the projection of $\vec{u}$ on $\vec{v}$.

$$\vec{u}_{\vec{v}} = \frac{\vec{u}\cdot\vec{v}}{|\vec{v}|^2}\vec{v} = \frac{(1\cdot 1 + 2\cdot 1)}{(\sqrt{1+1})^2}(\vec{i}+\vec{j}) = \frac{3}{2}(\vec{i}+\vec{j}) = \frac{3}{2}\vec{i} + \frac{3}{2}\vec{j} = \vec{u}_{\vec{v}} \parallel \vec{v}$$

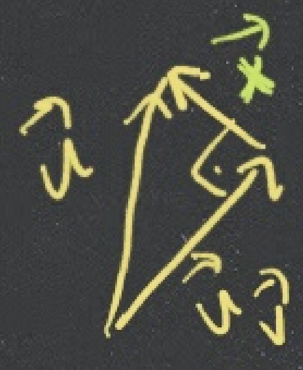


(2) Write $\vec{u} = \vec{i} + 2\vec{j}$ as a sum of two perpendicular vectors using $\vec{u}_{\vec{v}} = \frac{3}{2}\vec{i} + \frac{3}{2}\vec{j}$.

$\vec{u} = \vec{u}_{\vec{v}} + \vec{x}$ where $\vec{u}_{\vec{v}} \perp \vec{x}$

$$\Downarrow$$

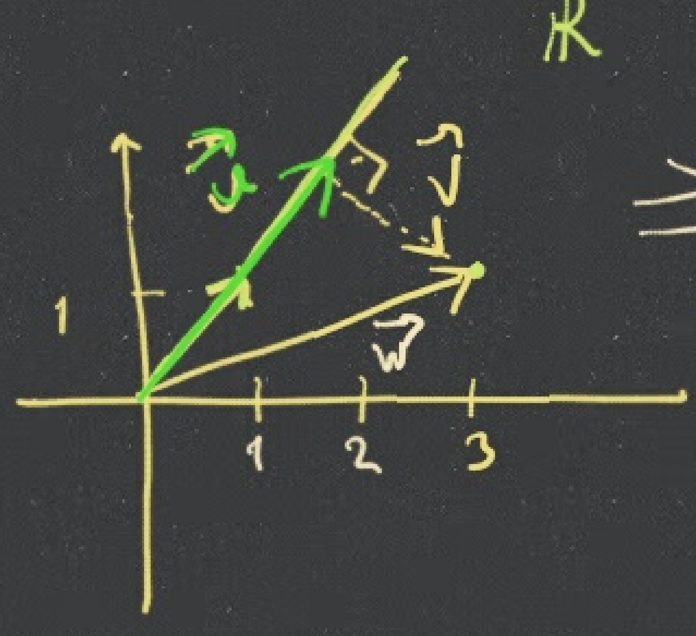
$$\vec{x} = \vec{u} - \vec{u}_{\vec{v}} = \vec{i} + 2\vec{j} - \left(\frac{3}{2}\vec{i} + \frac{3}{2}\vec{j}\right) = -\frac{1}{2}\vec{i} + \frac{1}{2}\vec{j}$$



$$\vec{u} = \vec{i} + 2\vec{j} = \underbrace{\frac{3}{2}\vec{i} + \frac{3}{2}\vec{j}}_{\vec{u}_{\vec{v}}} + \underbrace{-\frac{1}{2}\vec{i} + \frac{1}{2}\vec{j}}_{=\vec{x}}$$

Example: Express the vector $\vec{w} = 3\vec{i} + \vec{j}$ as sum $\vec{u} + \vec{v}$ where $\vec{u}$ is perpendicular to $\vec{v}$ and $\vec{u}$ is parallel to $\vec{i} + \vec{j}$.

$(\vec{w}_{\vec{u}} = \vec{w}_{\vec{i}+\vec{j}})$



$\Rightarrow$ Let $\vec{u} = \vec{w}_{\vec{i}+\vec{j}}$ and $\boxed{\vec{v} := \vec{w} - \vec{u}}$; then $\boxed{\vec{u} + \vec{v} = \vec{w}}$ and $\vec{u} \perp \vec{v}$.

$$\vec{u} = \frac{\vec{w}\cdot(\vec{i}+\vec{j})}{|\vec{i}+\vec{j}|^2}(\vec{i}+\vec{j}) = \frac{(3\vec{i}+\vec{j})\cdot(\vec{i}+\vec{j})}{(\sqrt{2})^2}(\vec{i}+\vec{j}) = \frac{3+1}{2}(\vec{i}+\vec{j}) = \boxed{2\vec{i} + 2\vec{j} = \vec{u}} \;//$$

$$\Rightarrow \vec{v} = \vec{w} - \vec{u} = 3\vec{i} + \vec{j} - (2\vec{i} + 2\vec{j}) = \boxed{\vec{i} - \vec{j} = \vec{v}} \;//$$

$\Rightarrow 2\vec{i} + 2\vec{j} + \vec{i} - \vec{j} = 3\vec{i} + \vec{j} = \vec{w}$ ✓

Alternative solution without using projection:

Since $\vec{u} \parallel \vec{i} + \vec{j}$, write $\vec{u} = t\vec{i} + t\vec{j}$ for some $t \in \mathbb{R}$, $t \neq 0$.

write $\vec{v} = x\vec{i} + y\vec{j}$ for some $x, y \in \mathbb{R}$.

We know $\vec{u} \perp \vec{v}$, that means $0 = \vec{u}\cdot\vec{v} = (t\vec{i} + t\vec{j})\cdot(x\vec{i} + y\vec{j}) = tx + ty = t(x+y)$, $t \neq 0$

$x + y = 0 \Rightarrow x = -y \Rightarrow y = -x$

$$\text{Also, } \vec{w} = 3\vec{i} + \vec{j} = \vec{u} + \vec{v} = (t\vec{i} + t\vec{j}) + (x\vec{i} - x\vec{j}) = (t+x)\vec{i} + (t-x)\vec{j}$$

$$\Rightarrow \begin{array}{r} 3 = t + x \;* \\ +\; 1 = t - x \\ \hline 4 = 2t \end{array} \Rightarrow t = 2 \overset{*}{\Rightarrow} x = 1 \Rightarrow y = -1 \Rightarrow \boxed{\vec{v} = \vec{i} - \vec{j}} \;//$$

and $\boxed{\vec{u} = 2\vec{i} + 2\vec{j}}$ //

What we did in this section for vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$ are still true in $\mathbb{R}^n$ for any $\underline{n \geq 2}$.

including vector addition, scalar mult.

dot product of vectors

projection of one vector to another one.

Every vector $\vec{v}$ in $\mathbb{R}^n$ can be written as a linear combination of $\vec{e}_i$'s where $\vec{e}_i = \overrightarrow{0e_i}$

$e_i = (0, \dots, 0, 1, 0, \dots, 0) \in \mathbb{R}^n$ for $\underline{i = 1, \dots, n}$,

If $\vec{v}$ is the vector with components $(v_1, \dots, v_n)$

then $\vec{v} = v_1\vec{e}_1 + \dots + v_n\vec{e}_n$

and $\vec{u}$ is the vector with components $(u_1, \dots, u_n)$

$\vec{u} = u_1\vec{e}_1 + \dots + u_n\vec{e}_n$.

Then $\vec{u} \cdot \vec{v} = u_1 v_1 + \dots + u_n v_n$

and $|\vec{u}| = \sqrt{\vec{u} \cdot \vec{u}} = \sqrt{u_1^2 + \dots + u_n^2}$

and $\vec{u} \perp \vec{v} \iff \vec{u} \cdot \vec{v} = 0$

and $\vec{u} \cdot \vec{v} = |\vec{u}| \cdot |\vec{v}| \cdot \cos(\theta)$; $\theta = \arccos\left(\frac{\vec{u} \cdot \vec{v}}{|\vec{u}||\vec{v}|}\right)$.

$\vec{u}_{\vec{v}} = \frac{\vec{u} \cdot \vec{v}}{|\vec{v}|^2} \vec{v}$ is <u>the vector projection of $\vec{u}$ on $\vec{v}$</u>.

& $|\vec{u}_{\vec{v}}| = \frac{\vec{u} \cdot \vec{v}}{|\vec{v}|} = \frac{|\vec{u}|\cancel{|\vec{v}|}\cos(\theta)}{\cancel{|\vec{v}|}} = |\vec{u}|\cos(\theta)$.

End of 10.2.