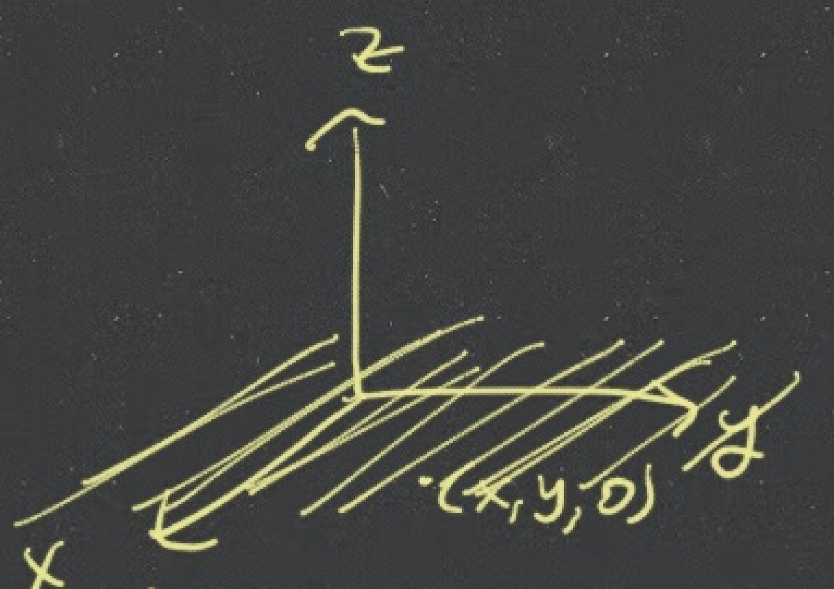
10.1 . Part 3.

Euclidean n-space $\equiv$ n-dimensional space , $n \geq 3$

$$\mathbb{R}^n = \mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R} \quad \text{n-copies of } \mathbb{R}$$
$$= \{ (x_1, x_2, \dots, x_n) \mid x_i \in \mathbb{R} \}$$

$P, Q \in \mathbb{R}^n$ with $P = (x_1, x_2, \dots, x_n)$, $Q = (y_1, y_2, \dots, y_n)$

Then $|PQ| =$ distance between P & $Q = \sqrt{(y_1 - x_1)^2 + \cdots + (y_n - x_n)^2}$



An $(n-1)$-dimensional set of points in $\mathbb{R}^n$ satisfying $x_n = 0$ is called a hyperplane, by analogy with the plane $z=0$ in $\mathbb{R}^3$, xy-plane

Describing Sets in the plane $\mathbb{R}^2$, in $\mathbb{R}^3$, in n-space

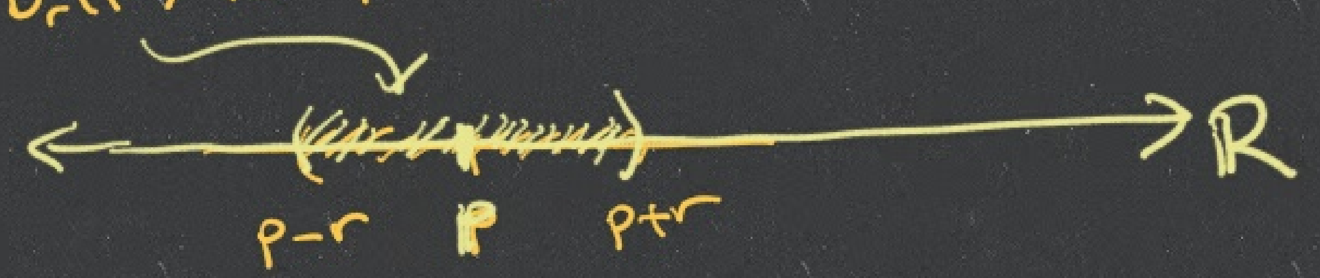
TOPOLOGY OF $\mathbb{R}^n$, OPEN SETS , CLOSED SETS, COMPLEMENT OF A SET

A neighbourhood of a point P in $\mathbb{R}^n \equiv B_r(P) = \{ Q \in \mathbb{R}^n \mid |PQ| < r \}$ for some $r > 0$.

Examples:

1) $n=1$, $p \in \mathbb{R}$, then $B_r(p) = (p-r, p+r)$

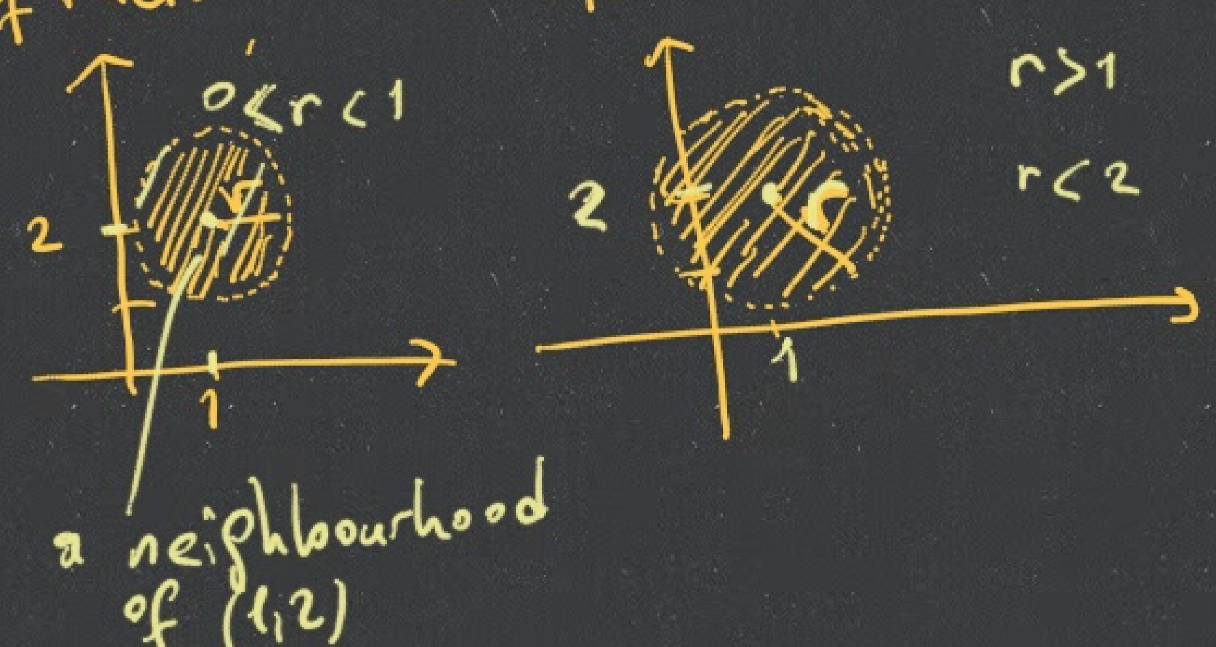
$B_r(P)$ is the open interval



$$B_r(P) = \{ x \in \mathbb{R} \mid p-r < x < p+r \}$$
$$\shortparallel$$
$$\{ x \in \mathbb{R} \mid |x - p| < r \}$$

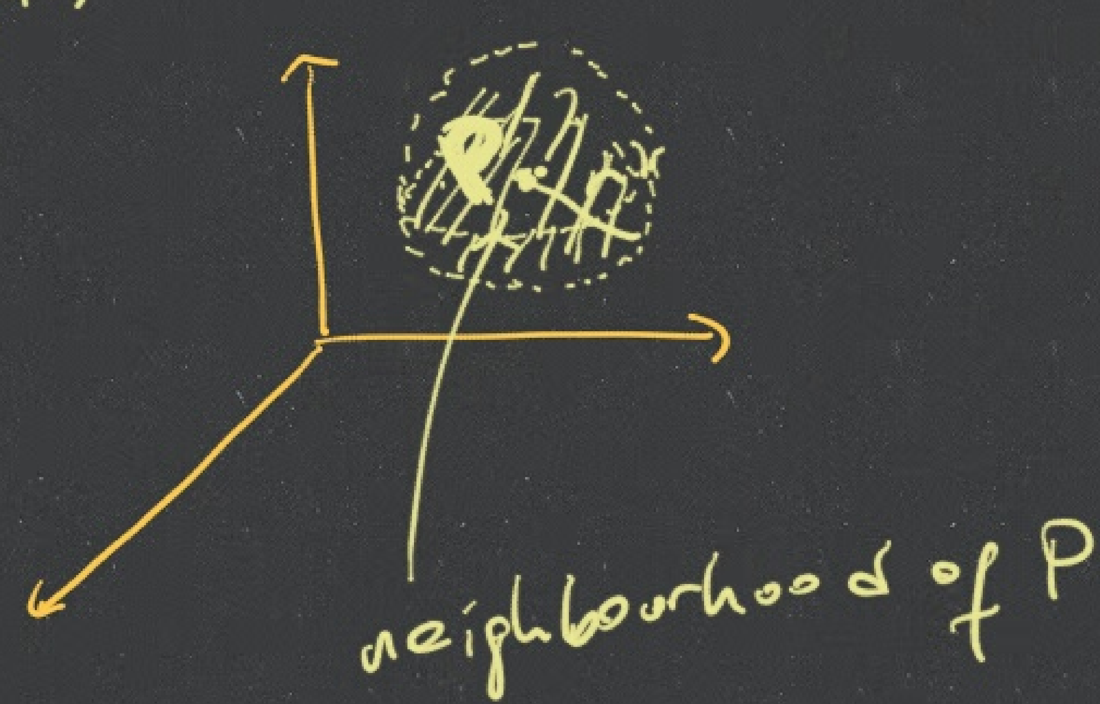
2) $n=2$, $P = (1,2)$

$B_r(P) \equiv$ open disk centered at P of radius $r \equiv \{ (x,y) \mid \sqrt{(x-1)^2 + (y-2)^2} < r \}$



3) $n=3$, $P = (a, b, c)$,

$$B_r(P) = \{ Q \in \mathbb{R}^3 \mid |PQ| < r \}$$
$$= \{ (x,y,z) \in \mathbb{R}^3 \mid \sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2} < r \}$$



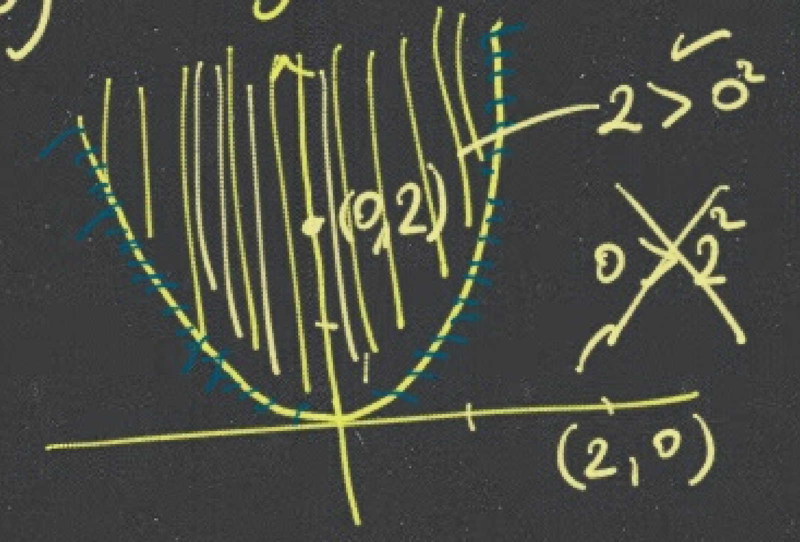
A set $S \subset \mathbb{R}^n$ is OPEN if every point of S has a neighbourhood contained in S.

Examples of OPEN sets:

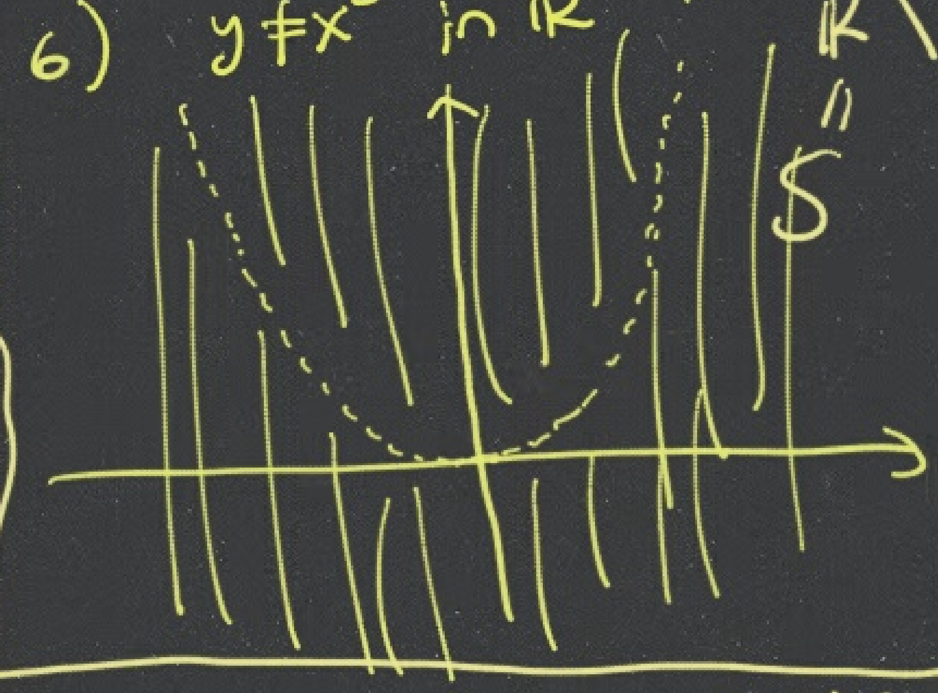
1) $B_r(P)$ | 2) $\mathbb{R}^n$ | 3) ϕ

4) $x>0,\ y>0$ in $\mathbb{R}^2$

5) $y > x^2$ in $\mathbb{R}^2$



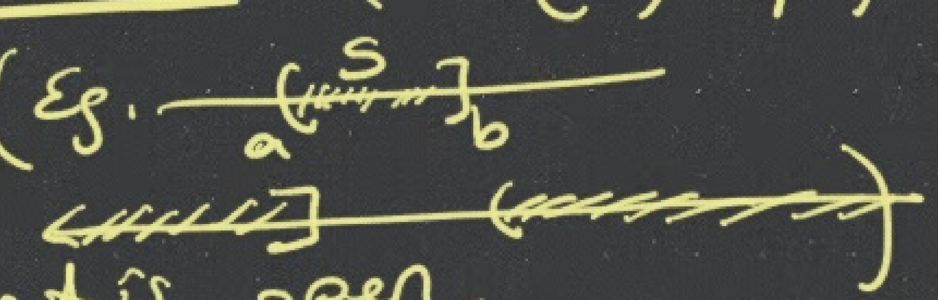
6) $y \neq x^2$ in $\mathbb{R}^2$ $\quad \mathbb{R}^2 \setminus$ parabola $= S$



• The COMPLEMENT, S^c, of a set S in $\mathbb{R}^n$ is $\mathbb{R}^n \setminus S$. (Eg: $(\mathbb{R}^n)^c = \phi$)

$S^c = \mathbb{R}^n \setminus S \equiv$ points of $\mathbb{R}^n$ which are not in S.

(Eg.

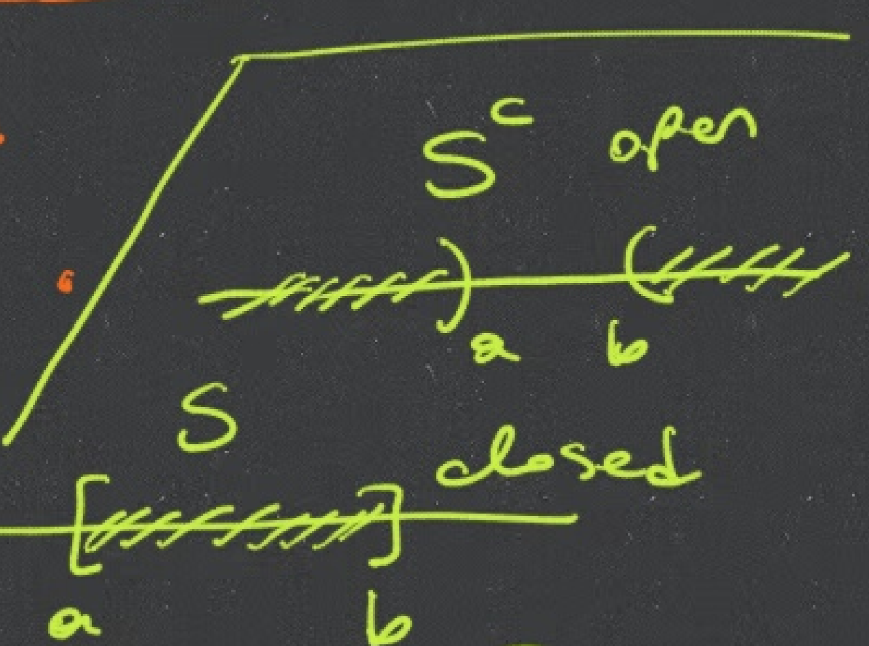


)

• A set is called CLOSED if its complement is open.

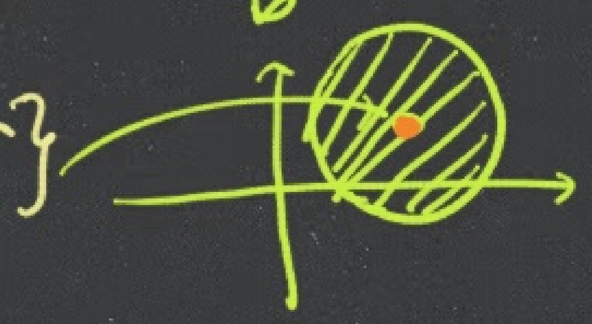
Examples:

1) $\mathbb{R}^n$ is both open and closed $\mathbb{R}^n \setminus \mathbb{R}^n = \phi = (\mathbb{R}^n)^c$.



2) Closed intervals in $\mathbb{R}$ are closed sets.

3) A closed disk in $\mathbb{R}^2 \equiv \{(x,y)\in\mathbb{R}^2 \mid (x-a)^2+(y-b)^2 \leq r\}$ (a,b)



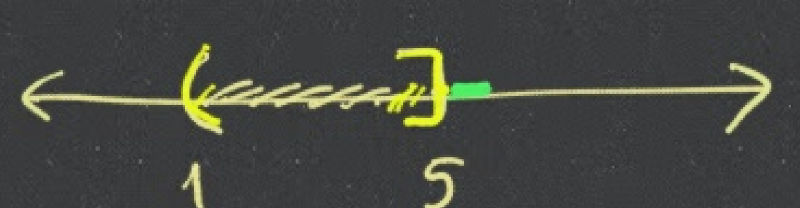
4) A closed disk in $\mathbb{R}^n \equiv \{(x_1,\dots,x_n)\in\mathbb{R}^n \mid (x_1-a_1)^2+\dots+(x_n-a_n)^2 \leq r\}$

5) $\mathbb{R}^n \setminus \boxed{B_r(P)} = \{(x_1,\dots,x_n)\in\mathbb{R}^n \mid (x_1-a_1)^2+\dots+(x_n-a_n)^2 \geq r\}$

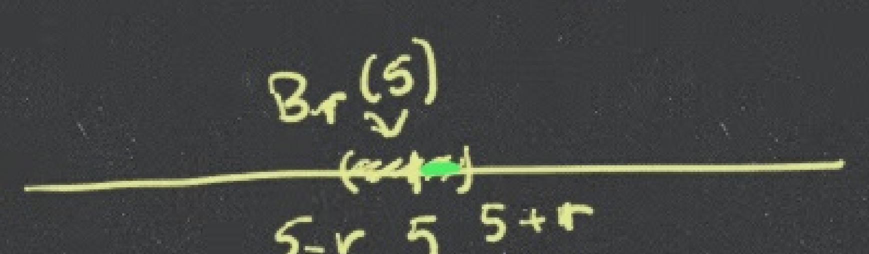
$P = (a_1,\dots,a_n)$ $\quad n=1$

6) A set can be neither open, nor closed.

. $n=1$,



$S = (1,5]$

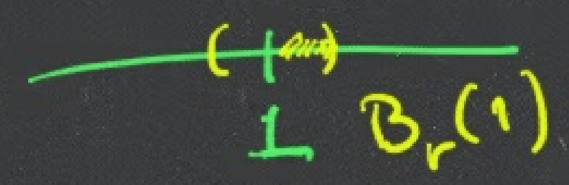


$B_r(5) \cap S^c \neq \phi$ for any $r>0$.

$\Rightarrow$ S is not open.

$\Rightarrow S^c$ is not open

$\Rightarrow$ S cannot be closed

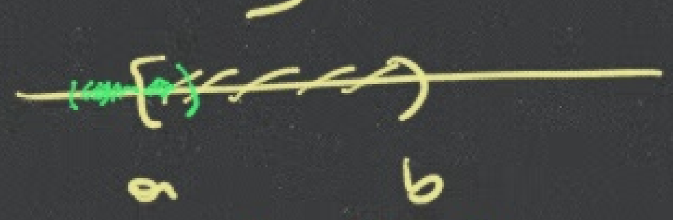


Let S be a set in $\mathbb{R}^n$.
A point $P \in S$ is called a boundary point of S if every neighbourhood of P contains points from S and S^c.
The set of boundary points of S is denoted by bdry(S) or ∂S

1) S

$\partial(S) = \phi$

2) S

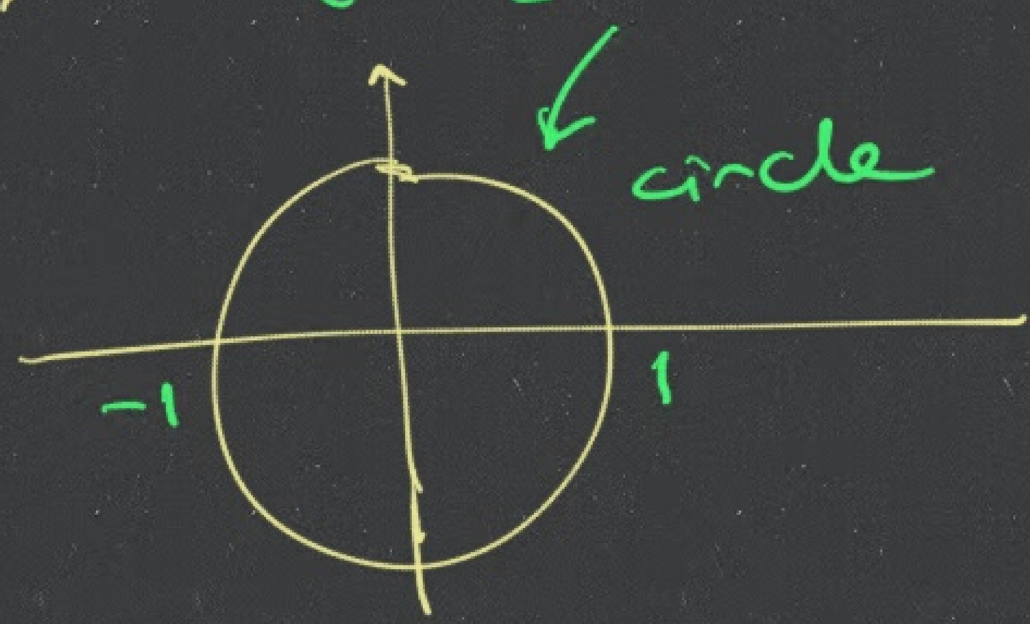


$\partial(S) = \{a\}$

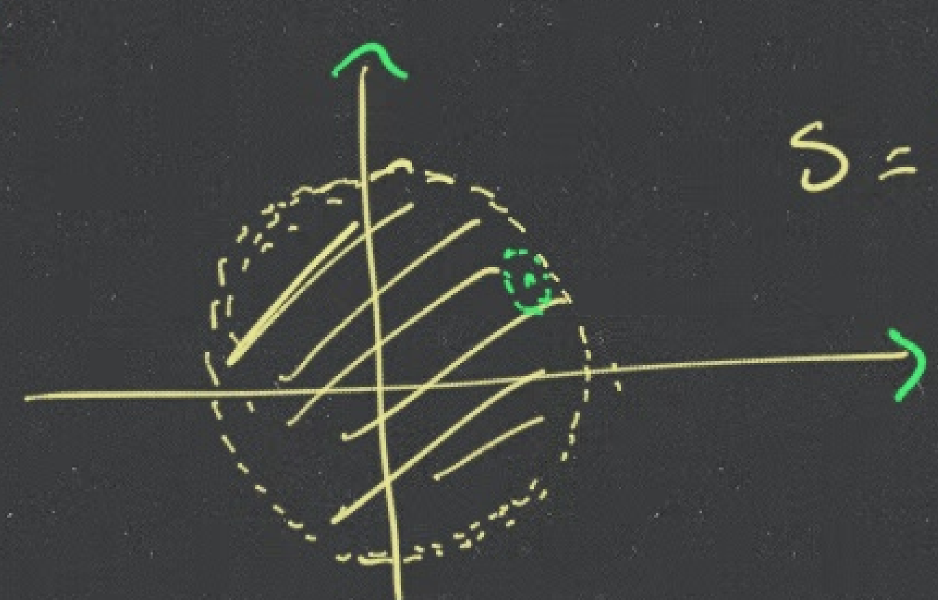
3) S

$\partial(S) = \{a, b\}$

4) $S = \{(x,y) \mid x^2+y^2 \leq 1\} \Rightarrow \partial(S) = \{(x,y) \mid x^2+y^2 = 1\}$



5) $S = \{(x,y) \mid x^2+y^2 < 1\} \Rightarrow \partial(S) = \phi$.



- A point P is called an interior point of S if $P \in S$ but $P \notin \partial(S)$. The set of such points is denoted by int(S)

int$(S) = S \setminus \partial(S)$.

- A point P is called an exterior point of S if $P \in S^c$ but $P \notin \partial(S)$. The set of such points is denoted by ext(S).

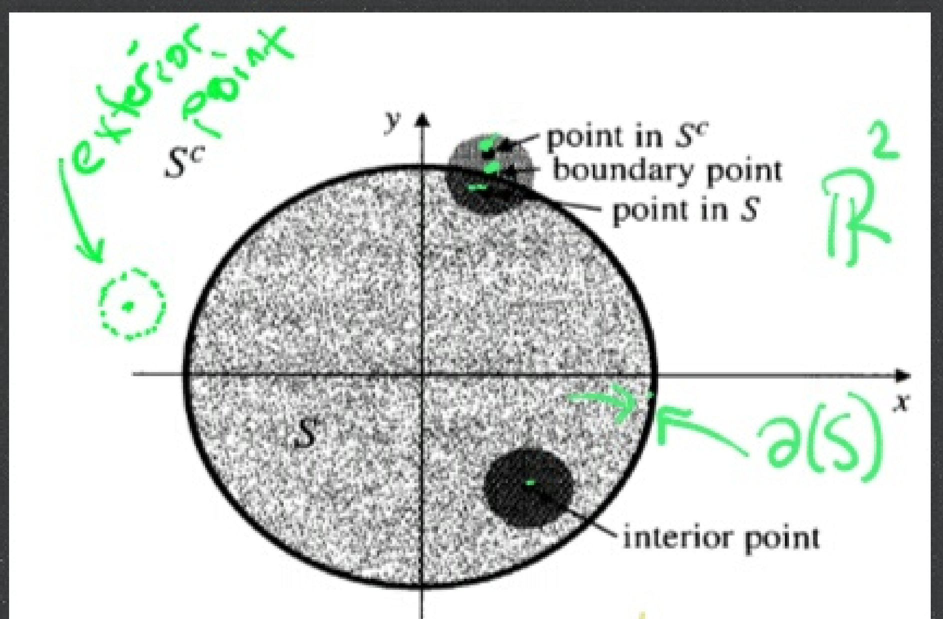


Figure 10.10 The closed disk S consisting of points $(x, y) \in \mathbb{R}^2$ that satisfy $x^2 + y^2 \leq 1$. Note the shaded neighbourhoods of the boundary point and the interior point.
bdry(S) is the circle $x^2 + y^2 = 1$
int(S) is the open disk $x^2 + y^2 < 1$
ext(S) is the open set $x^2 + y^2 > 1$

ext$(S) = S^c \setminus \partial(S)$

- A set S is an open set $\Leftrightarrow S = \text{int}(S)$
- " " is a closed set $\Leftrightarrow S^c = \text{ext}(S)$

If $\partial(S) = \phi$, then $S = \text{int}(S)$ and $S^c = \text{ext}(S)$

then S is both open and closed.