10.1. Part 3.

Euclidean $n$-space $\equiv n$-dimensional space, $n \geq 3$

\[ \mathbb{R}^n = \mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R} \text{ } n \text{-copies of } \mathbb{R} \]

$P, Q \in \mathbb{R}^n$ with $P = (x_1, x_2, \ldots, x_n)$, $Q = (y_1, y_2, \ldots, y_n)$

Then $|PQ| = \text{distance between } P \text{ and } Q = \sqrt{(x_1-y_1)^2 + \cdots + (x_n-y_n)^2}$

An $(n-1)$-dimensional set of points in $\mathbb{R}^n$ satisfying $x_i = 0$ is called a hyperplane, by analogy with the plane $z = 0$ in $\mathbb{R}^3$, $xy$-plane.

Describing sets in the plane $\mathbb{R}^2$ in $\mathbb{R}^n$, in $n$-space.

Topology of $\mathbb{R}^n$, open sets, closed sets, complement of a set.

A neighbourhood of a point $P$ in $\mathbb{R}^n \equiv \mathbb{B}_r(P) = \{ Q \in \mathbb{R}^n | |PQ| < r \}$ for some $r > 0$.

Examples:

1) $n = 1$, $P \in \mathbb{R}$, then $\mathbb{B}_r(P) = (P-r, P+r)$

\[ \mathbb{B}_r(P) = \{ x \in \mathbb{R} | r < x < P + r \} \]

\[ \{ x \in \mathbb{R} | |x-P| < r \} \]

2) $n = 2$, $P = (1, 2)$

\[ B_r(P) = \{ (x, y) | \sqrt{(x-1)^2 + (y-2)^2} < r \} \]

3) $n = 3$, $P = (a, b, c)$,

\[ \mathbb{B}_r(P) = \{ (x, y, z) \in \mathbb{R}^3 | |PQ| < r \} \]

\[ = \{ (x, y, z) \in \mathbb{R}^3 | \sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2} < r \} \]

neighbourhood of $P$
A set $S \subseteq \mathbb{R}^n$ is **open** if every point of $S$ has a neighborhood contained in $S$.

**Examples of open sets:**

1) $B_r(P)$

2) $\mathbb{R}^n$

3) $\emptyset$

4) $x > 0, y > 0$ in $\mathbb{R}^2$

5) $y > x^2$ in $\mathbb{R}^2$

6) $y = x^2$ in $\mathbb{R}^2$

$\mathbb{R} \setminus \{r\}$ is an example of a set that is not open.

The **complement** $S^c$ of a set $S$ in $\mathbb{R}^n$ is $\mathbb{R}^n \setminus S$. (E.g., $(\mathbb{R})^c = \emptyset$)

$S^c = \mathbb{R}^n \setminus S$ represents points of $\mathbb{R}^n$ which are not in $S$.

**A set is called closed if its complement is open.**

**Examples:**

1) $\mathbb{R}$ is both open and closed.

2) Closed intervals in $\mathbb{R}$ are closed sets.

3) A closed disk in $\mathbb{R}^2 \equiv \{(x,y) \in \mathbb{R}^2 \mid (x-a)^2 + (y-b)^2 \leq r^2\}$

4) A closed disk in $\mathbb{R}^n \equiv \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid (x_1-a)^2 + \ldots + (x_n-a)^2 \leq r^2\}$

5) $\mathbb{R}^n \setminus \overline{B_r(P)} = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid (x_1-a)^2 + \ldots + (x_n-a)^2 > r^2\}$

A set can be neither open, nor closed.

$n=1, \quad S = (1,5]$

$\overline{B_r(5)} = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid (x_1-a)^2 + \ldots + (x_n-a)^2 = r^2\}$

$B_r(5) \cap S^c \neq \emptyset$ for any $r > 0$.

$\Rightarrow S$ is not open.

$S^c \Rightarrow S^c$ is not open.

$S \not\subseteq S^c \Rightarrow S$ cannot be closed.
Let $S$ be a set in $\mathbb{R}^n$. A point $P \in S$ is called a boundary point of $S$ if every neighbourhood of $P$ contains points from $S$ and $S^c$. The set of boundary points of $S$ is denoted by $\partial S$ or $\partial(S)$.

1) $S = \{a, b\}$, $\partial(S) = \emptyset$
2) $S = \{a\}$, $\partial(S) = \{a\}$
3) $S = \{a, b\}$, $\partial(S) = \{a, b\}$

4) $S = \{(x, y) \mid x^2 + y^2 \leq 1\}$, $\partial(S) = \{(x, y) \mid x^2 + y^2 = 1\}$

5) $S = \{(x, y) \mid x^2 + y^2 < 1\}$, $\partial(S) = \emptyset$.

A point $P$ is called an interior point of $S$ if $P \in S$ but $P \notin \partial(S)$. The set of such points is denoted by $\text{int}(S)$.

A point $P$ is called an exterior point of $S$ if $P \in S^c$ but $P \notin \partial(S)$. The set of such points is denoted by $\text{ext}(S)$.

$\text{int}(S) = S \setminus \partial(S)$

$\text{ext}(S) = S^c \setminus \partial(S)$

A set $S$ is an open set if $S = \text{int}(S)$, and it is a closed set if $S = \text{ext}(S)$. If $\partial(S) = \emptyset$, then $S = \text{int}(S)$ and $S^c = \text{ext}(S)$, and $S$ is both open and closed.