$$\sum_{n=0}^{\infty} a_n (x-c)^n$$ Suppose $R > 0$, and

$$f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n \quad \text{on } (c-R, c+R).$$

Question: How do the a_n relate to $f(x)$?

$f^{(0)}(c) = f(c) = a_0$ $\quad \left[f(x) = a_0 + a_1(x-c) + a_2(x-c)^2 + \cdots \right]$

$f'(x) = a_1 + 2a_2(x-c) + 3a_3(x-c)^2 + \cdots$

$f'(c) = a_1$

$f''(x) = 2a_2 + 6a_3(x-c) +$ higher order terms

$f''(c) = 2a_2$

$f'''(x) = 6a_3 +$ higher order terms

$f'''(c) = 6a_3$

$$\boxed{\begin{aligned} (x^n)' &= n x^{n-1} \\ (x^n)'' &= n(n-1) x^{n-2} \\ (x^n)''' &= n(n-1)(n-2) x^{n-3} \\ &\vdots \end{aligned}}$$

In general, $a_n = \dfrac{f^{(n)}(c)}{n!}$

If $f(x)$ has derivatives of all orders at $x=c$, then

$$\sum^{\infty} \frac{f^{(n)}(c)}{} (x-c)^n$$

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$$

is called the Taylor series of f about c.

If $c=0$, it is called the Maclaurin series of $f(x)$.

In general, $f(x) \rightsquigarrow \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$ may not have $R>0$, and even if it does, may not converge to $f(x)$ except at $x=c$!

In this course, this problem does not concern us!

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ex: Write the Maclaurin series for $f(x)=e^x$.

$$f^{(n)}(x)=e^x \qquad f^{(n)}(\underbrace{0}_{})=1$$

c=0 for Maclaurin series

$$\sum_{n=0}^{\infty} \frac{1}{n!}x^n = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

In fact, $\boxed{e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad \text{for all } x.}$

ex: Write Maclaurin series for $e^{x^3/4}$.

$$e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!} \xrightarrow[\text{by } \frac{x^3}{4}]{\text{replace } x} e^{x^3/4} = \sum_{n=0}^{\infty}\frac{\left(\frac{x^3}{4}\right)^n}{n!}$$

$$= \sum_{n=0}^{\infty}\frac{x^{3n}}{4^n \cdot n!} \quad \text{for all } x.$$

ex: Write Maclaurin series for $\underbrace{\dfrac{e^{x^2}-1}{x}}_{f(x)}$.

$f(0) = \lim_{x \to 0} \dfrac{e^{x^2}-1}{x}$

$$e^{x^2} = \sum_{n=0}^{\infty} \frac{(x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!} \quad \text{for all } x.$$

$$e^{x^2} = \underset{n=0}{1} + \left[\underset{n=1}{x^2} + \underset{n=2}{\frac{x^4}{2!}} + \cdots \right]$$

$$e^{x^2} - 1 = \left[\sum_{n=1}^{\infty} \frac{x^{2n}}{n!} \right] \quad \text{for all } x.$$

$$\frac{e^{x^2}-1}{x} = \frac{1}{x} \cdot \sum_{n=1}^{\infty} \frac{x^{2n}}{n!} = \sum_{n=1}^{\infty} \frac{x^{2n-1}}{n!} \quad \text{for all } x.$$

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ex: Find Maclaurin series for $f(x) = \sin x$.

$f'(x) = \cos x \qquad f''(x) = -\sin x \qquad f'''(x) = -\cos x \qquad f^{(4)}(x) = \sin x$

$f(0) = 0 \qquad f'(0) = 1 \qquad f''(0) = 0 \qquad f'''(0) = -1 \qquad f^{(4)}(0) = 0$

$f^{(5)}(0) = 1 \qquad f^{(6)}(0) = 0 \qquad f^{(7)}(0) = -1 \qquad f^{(8)}(0) = 0.$

The Maclaurin series is: $x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!} + \cdots$

$$\boxed{\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \quad \text{for all } x.}$$

$$\boxed{\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \cdots \quad \text{for all } x}$$

$$\boxed{e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \quad \text{for all } x}$$

$$\sum_{n=0} \quad n! \qquad 2! \quad 3!$$

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ex: Write Maclaurin series for $\dfrac{1-\cos(x^2)}{x^3}$.

$$\cos(x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n (x^2)^{2n}}{(2n)!} = 1 - \frac{(x^2)^2}{2!} + \frac{(x^2)^4}{4!} - \dots \quad \text{for all } x$$

$$1-\cos(x^2) = 1 - \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n}}{(2n)!} = \cancel{1} - \Big(\underset{n=0}{\cancel{1}} - \underset{n=1}{\frac{x^4}{2!}} + \underset{n=2}{\frac{x^8}{4!}} - \dots\Big)$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^{4n}}{(2n)!} = \frac{x^4}{2!} - \frac{x^8}{4!} + \dots \quad \text{for all } x$$

$$\underbrace{\frac{1-\cos(x^2)}{x^3}}_{f(x)} = \frac{1}{x^3} \cdot \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^{4n}}{(2n)!} = \frac{1}{x^3}\Big(\frac{x^4}{2!} - \frac{x^8}{4!} - \dots\Big)$$

$$= \frac{x}{2!} - \frac{x^5}{4!} \dots = \boxed{\sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^{4n-3}}{(2n)!} \quad \text{for all } x}$$

$$\left(f(0) = \lim_{x \to 0} \frac{1-\cos(x^2)}{x^3} \right)$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 \dots \quad \text{for } x \in (-1, 1).$$

$$\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} (n+1) x^n = 1 + 2x + 3x^2 + 4x^3 + \dots \quad \text{for } x \in (-1, 1)$$

$$\ln(1+x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} \quad \text{for } x \in (-1, 1]$$

$$\tan^{-1} x = \sum \frac{(-1)^n x^{2n+1}}{\dots} = x - \frac{x^3}{\dots} + \frac{x^5}{\dots} - \frac{x^7}{\dots} \dots \quad \text{for } x \in [-1, 1]$$

$$\tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} \cdots \text{ for } x \in [-1,1]$$

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!} (x-c)^k$$ Taylor polynomials

$$E_n(x) = f(x) - P_n(x)$$

$$\underbrace{f(x)}_{\text{actual function value}} = \underbrace{P_n(x)}_{\text{approximation}} + \underbrace{E_n(x)}_{\text{error term}}$$

Taylor's Theorem: If the $n+1^{st}$ derivative of f exists on an interval containing c and x, then

$$f(x) = P_n(x) + E_n(x) \qquad \text{(Taylor's formula)}$$

where $E_n(x) = \dfrac{f^{(n+1)}(s)}{(n+1)!}(x-c)^{n+1}$ for some s between c and x.

Ex: Use Taylor's Theorem to show that the Maclaurin series for $\sin x$ converges to $\sin x$.

$$\underbrace{f(x)}_{=\sin x} = \underbrace{P_n(x)}_{\text{partial sum of the Maclaurin series}} + E_n(x) \longrightarrow$$ We want to show:

$$\lim_{n\to\infty} E_n(x) = 0$$

$$|E_n(x)| = \frac{|f^{(n+1)}(s)|}{(n+1)!}|x|^{n+1} \leq \frac{1}{\underbrace{(n+1)!}}|x|^{n+1}$$

approaches 0
as $n \to \infty$.

So $\lim_{n \to \infty} E_n(x) = 0$.

So the Maclaurin series of $\sin x$ converges to $\sin x$ for all x.