$$\sum_{n=0}^{\infty} a_n x^n$$

<u>Thm</u>: $\sum_{n=0}^{\infty} a_n x^n$, $\sum_{n=0}^{\infty} b_n x^n$ R_a, R_b, $c \in \mathbb{R}$.

i) $\sum_{n=0}^{\infty} (c a_n) x^n$ has radius of convergence R_a

and $\sum_{n=0}^{\infty} (c a_n) x^n = c \cdot \left(\sum_{n=0}^{\infty} a_n x^n \right)$

when the series on the right converges

$(c \neq 0)$

ii) $\sum_{n=0}^{\infty} (a_n \mp b_n) x^n$ has $R \geq \min\{R_a, R_b\}$

and $\sum_{n=0}^{\infty} (a_n \mp b_n) x^n = \sum_{n=0}^{\infty} a_n x^n \mp \sum_{n=0}^{\infty} b_n x^n$

whenever both series on the right converge.

$$\left(\sum a_n x^n\right) \cdot \left(\sum b_n x^n\right) \neq \left(\sum a_n \cdot b_n \cdot x^n\right)$$

$$(a_0 + a_1 x + a_2 x^2 + \cdots) \cdot (b_0 + b_1 x + b_2 x^2 + \cdots)$$

$$= \underbrace{a_0 b_0}_{c} + \underbrace{(a_0 b_1 + a_1 b_0)}_{c} x + \underbrace{(a_0 b_2 + a_1 b_1 + a_2 b_0)}_{c_2} x^2$$

$$= \underbrace{a_0 b_0}_{c_0} + \underbrace{(a_0 b_1 + a_1 b_0)}_{c_1} x + \underbrace{(a_0 b_2 + a_1 b_1 + a_2 b_0)}_{c_2} x^2 + \cdots$$

$$\left(\sum_{n=0}^{\infty} a_n x^n\right) \cdot \left(\sum_{n=0}^{\infty} b_n x^n\right) = \sum_{n=0}^{\infty} c_n x^n, \quad \text{where}$$

(Cauchy product)

$$c_n = a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0 = \sum_{i=0}^{n} a_i \cdot b_{n-i}$$

$R_c \geq \min\{R_a, R_b\}$

---

ex: $$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n \quad \text{for } |x|<1$$

$(R=1)$

$$\frac{1}{(1-x)^2} = \frac{1}{1-x} \cdot \frac{1}{1-x} = \left(\sum_{n=0}^{\infty} x^n\right) \cdot \left(\sum_{n=0}^{\infty} x^n\right) = \sum_{n=0}^{\infty} c_n x^n$$

$a_n = 1$ $\quad$ $b_n = 1$

$R_a = 1$ $\quad$ $R_b = 1$

where $c_n = \underbrace{a_0}_{} b_n + \underbrace{a_1}_{} b_{n-1} + \cdots + \underbrace{a_n}_{} \cdot b_0$ $\quad$ $0, 1, 2, \ldots, n$ $\quad$ $n+1$ of them

$c_n = 1 + 1 + \cdots + 1$

$c_n = n+1$

$$\Rightarrow \frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} (n+1) x^n \qquad R_c \geq 1.$$

(in fact, $R_c = 1$)

interval of convergence $= (-1, 1)$.

Convergence $= (-1, 1)$.

<u>Thm</u>: If $\sum_{n=0}^{\infty} a_n x^n$ converges to $f(x)$ on $(-R, R)$, $R > 0$, that is, $f(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots$,

then f is differentiable on $(-R, R)$

and $f'(x) = \sum_{n=1}^{\infty} n \cdot a_n \cdot x^{n-1} = a_1 + 2a_2 x + 3a_3 x^2 + \cdots$

if $x \in (-R, R)$.

Also, f is integrable over any closed subinterval of $(-R, R)$ and if $|x| < R$, then

$$\int_0^x f(t)\, dt = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1} = a_0 x + \frac{a_1 x^2}{2} + \frac{a_2 x^3}{3} + \cdots$$

<u>ex</u>: $\frac{1}{(1-x)^3}$. Find a power series for it in powers of x.

$$\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} (n+1) x^n \qquad R = 1 \qquad (-1, 1)$$

$$\frac{d}{dx}\left[\frac{1}{(1-x)^2}\right] = \frac{-2}{(1-x)^3}$$

$dx \; [\qquad\qquad (1-x)^3$

$$\frac{1}{(1-x)^3} = -\frac{1}{2}\cdot\frac{d}{dx}\left[\frac{1}{(1-x)^2}\right] = -\frac{1}{2}\cdot\frac{d}{dx}\left[\sum_{n=0}^{\infty}(n+1)\cdot x^n\right]$$

$$= -\frac{1}{2}\cdot\sum_{n=0}^{\infty}\frac{d}{dx}\Big((n+1)\,x^n\Big) = -\frac{1}{2}\sum_{n=0}^{\infty}(n+1)\,n\,x^{n-1}$$

$$= -\frac{1}{2}\sum_{n=1}^{\infty}(n+1)\,n\,x^{n-1} = -\frac{1}{2}\sum_{n=0}^{\infty}(n+2)(n+1)\,x^n$$

down from 1 to 0

replace n by $n+1$

$R = 1$

interval: $(-1, 1)$

---

ex: Find a power series representation for $\ln(1+x)$ in powers of x.

$$\frac{d}{dx}\ln(1+x) = \frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty}(-x)^n$$

$$= \sum_{n=0}^{\infty}(-1)^n x^n$$

$R = 1$

interval: $(-1, 1)$

$-1 \quad 0 \quad 1$

$$\ln(1+x) = \int_0^x \frac{1}{1+t}\,dt = \sum_{n=0}^{\infty}\left(\int_0^x (-1)^n t^n\,dt\right)$$

$$= \sum^{\infty}(-1)^n \cdot x^{n+1} \qquad x \qquad x^2 \qquad x^3 \qquad x^4$$

$$\sum_{n=0} \frac{}{n+1} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} \cdots$$

for $-1 < x < 1$

$\ln(1+x)$ is continuous at $x=1$. By Abel's Theorem, the series converges to $\ln(1+x)$ at $x=1$ as well.

$$\overbrace{\ln(1+x)}^{\text{cont. on } (-1,1]} = \overbrace{\sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}}^{\text{cont. on } (-1,1]} \quad \text{for} \quad -1 < x \leq 1$$

If $x=1$: $$\ln 2 = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \cdots$$

(alternating harmonic series)!

Abel's Theorem: The sum of a power series is a continuous function on the interval of convergence.

ex: Find a power series representation for $\tan^{-1}x$ in powers of x.

$$\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2} = \frac{1}{1-(-x^2)}$$

$$= \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

$R = 1$

interval: $(-1,1)$

$n=0 \qquad n=0$

$\left(\text{If } x \in (-1,1),\ -x^2 \in (-1,1)\right)$

$$\tan^{-1} x = \int_0^x \frac{dt}{1+t^2} = \int_0^x \left(\sum_{n=0}^{\infty} (-1)^n t^{2n} \right) dt$$

$$= \sum_{n=0}^{\infty} \left(\int_0^x (-1)^n t^{2n}\, dt \right) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

$$= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots$$

$R = 1$, interval of convergence: $[-1, 1]$

(Homework: Check that the series converges for $x = \mp 1$. Hint: Use AST.)

$$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \quad \text{for } -1 \le x \le 1.$$

$$\tan^{-1} 1 = \frac{\pi}{4} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} \cdots$$