$$\sum_{n=0}^{\infty} \underbrace{a_n(x-c)^n}_{\text{general term}} = a_0 + a_1(x-c) + a_2(x-c)^2 + \dots$$

power series $\begin{cases} \text{in powers of } x-c \\ \text{about } c \end{cases}$

c : center of convergence

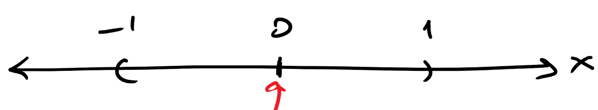
a_n: coefficients of the power series

ex: $\underbrace{1}_{a} + \underbrace{x}_{ar} + \underbrace{x^2}_{ar^2} + \underbrace{x^3}_{ar^3} + x^4 + \dots$ $\quad r = x$ power series in powers of $x = x - 0$ $(c=0)$

(in notation of 9.2)

this series $\begin{cases} \text{converges to } \dfrac{1}{1-x} \left(\dfrac{a}{1-r}\right) \text{ if } |x|<1 \\ \text{diverges if } |x| \geq 1 \end{cases}$ $\quad R = 1$

interval of convergence : $(-1, 1) = \{x \in \mathbb{R} \mid |x| < 1\}$



center of the interval

ex: $\displaystyle\sum_{n=1}^{\infty} \frac{(x-5)^n}{n \cdot 2^n}$. Determine the interval of convergence.

Start with Ratio Test to check for absolute convergence

$\displaystyle\sum_{n=1}^{\infty} \left| \frac{(x-5)^n}{n \cdot 2^n} \right|$. Apply Ratio Test

$$\sum_{n=1} \left| \frac{(x-5)^n}{n \cdot 2^n} \right| . \quad \text{Apply Ratio Test}$$

$$\rho = \lim_{n\to\infty} \frac{\left| \dfrac{(x-5)^{n+1}}{(n+1)\cdot 2^{n+1}} \right|}{\left| \dfrac{(x-5)^n}{n \cdot 2^n} \right|} = \lim_{n\to\infty} |x-5| \cdot \frac{n \cdot \cancel{2^n}}{(n+1)\cdot \cancel{2^{n+1}}_{\,2}}$$

$$= \lim_{n\to\infty} \frac{|x-5|}{2} \cdot \left(\frac{n}{n+1} \right) = \frac{|x-5|}{2}$$

- If $\frac{|x-5|}{2} < 1$, or $|x-5| < 2$, the power series converges absolutely.

$$-2 < x-5 < 2$$
$$3 < x < 7$$

$R = 2$

$|x-5| > 2$

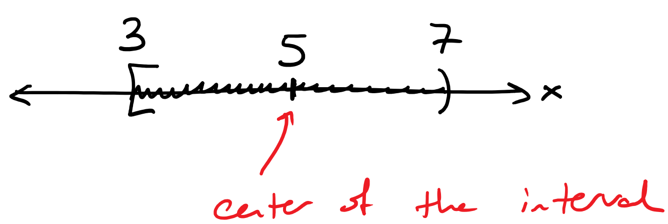
- If $x > 7$ or $x < 3$, the series diverges.

$x>7$: $\lim_{n\to\infty} \frac{\overbrace{(x-5)}^{>1}{}^n}{n \cdot 2^n} = \infty$ $\qquad$ $x<3$: $\lim_{n\to\infty} \frac{\overbrace{(x-5)}^{<-1}{}^n}{n \cdot 2^n}$ does not exist

- If $x = 7$: $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. If $x=3$: $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges (conditionally).

Interval of convergence for $\sum_{n=1}^{\infty} \frac{(x-5)^n}{n \cdot 2^n}$: $[3, 7)$

$(c = 5)$



Thm: $\sum_{n=0}^{\infty} a_n (x-c)^n$. Exactly one of the following holds:

1) the series converges only at $x=c$. $(R=0)$
2) the series converges at every real number x $(R=\infty)$

2) The series converges at every x ... $(-\infty, \infty)$

3) there is a positive $R > 0$ such that the series converges if $|x-c| < R$ and diverges if $|x-c| > \boxed{R}$
(the series may or may not converge at $x = c \mp R$)

The convergence is absolute except possibly at $x = c \mp R$ in case (3).

R: radius of convergence

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ex: $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$. Determine the radius and interval of convergence.

$c = 0$

Use Ratio Test to determine absolute convergence. $\left(\displaystyle\sum_{n=0}^{\infty} \left| \frac{x^n}{n!} \right| \right)$

$$\rho = \lim_{n\to\infty} \frac{\left| \frac{x^{n+1}}{(n+1)!} \right|}{\left| \frac{x^n}{n!} \right|} = \lim_{n\to\infty} \frac{|x|}{(n+1)} = 0 < 1$$

The power series $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges absolutely for all x.

$R = \infty$ Interval of convergence $= (-\infty, \infty)$

(Case 2 in Theorem)

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ex: $\displaystyle\sum_{n=0}^{\infty} n! \cdot x^n$. Determine the radius and the interval of convergence.

$c = 0$

(The series converges absolutely if $x = 0$.)

Assume $x \neq 0$:

$$\rho = \lim \frac{|(n+1)!\, x^{n+1}|}{\quad} = \lim (n+1) \cdot |x| = \infty$$

$$\rho = \lim_{n\to\infty} \frac{\qquad}{|n!\cdot x^n|} = \lim_{n\to\infty} (n+1)\cdot|x| = \infty$$

$\underline{x>0}$: $\lim\limits_{n\to\infty} n!\,x^n = \infty$ $\qquad$ $\underline{x<0}$: $\lim\limits_{n\to\infty} n!\,x^n$ does not exist

$\Longrightarrow$ If $x \neq 0$, the series $\sum\limits_{n=0}^{\infty} n!\,x^n$ diverges (n^{th} term test for divergence)

$R = 0$ $\qquad$ Interval of convergence: $\{0\} = \underbrace{[0,0]}_{\text{radius of this interval} = 0}$

Suppose $L = \lim\limits_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right|$ exists or is ∞. Then the power series $\sum\limits_{n=0}^{\infty} a_n (x-c)^n$ has radius of convergence $R = \dfrac{1}{L}$

(If $L=0$, $R=\infty$. If $L=\infty$, $R=0$.)