$$\sum_{n=1}^{\infty} a_n \quad \text{where} \quad a_n \geq 0 \quad (\text{ultimately})$$

$$\sum_{n=1}^{\infty} a_n \quad \text{where} \quad a_n \geq 0 \quad \text{or} \quad a_n \leq 0$$

$$\sum_{n=1}^{\infty} (-1)^n \cdot \frac{1}{n^2+1} = -\frac{1}{2} + \frac{1}{5} - \frac{1}{10} + \frac{1}{17} \cdots$$

<u>Def</u>: $\sum a_n$ is absolutely convergent (converges absolutely if $\sum |a_n|$ converges.

<u>ex</u>: $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n^3}$ converges absolutely since

$$\sum_{n=1}^{\infty} \left| (-1)^n \frac{1}{n^3} \right| = \sum_{n=1}^{\infty} \frac{1}{n^3} \quad (p\text{-series}, \quad p=3)$$

Converges.

<u>Thm</u>: If $\sum a_n$ converges absolutely, then it converges.

(Absolute convergence $\Longrightarrow$ convergence)

∞

ex: $\sum_{n=1} (-1)^n \frac{1}{n^3}$ converges because it converges absolutely.

Convergence does not imply absolute convergence.

ex: $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ (alternating harmonic series)

$= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \cdots$ converges. (by Alternating Series Test)

$+ \quad - \quad + \quad -$ (alternating series)

It does not converge absolutely:

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n} \quad \text{diverges!}$$

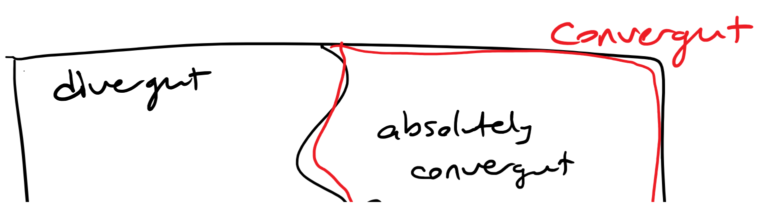
Def: If a series converges, but it does not converge absolutely, it is said to converge 1

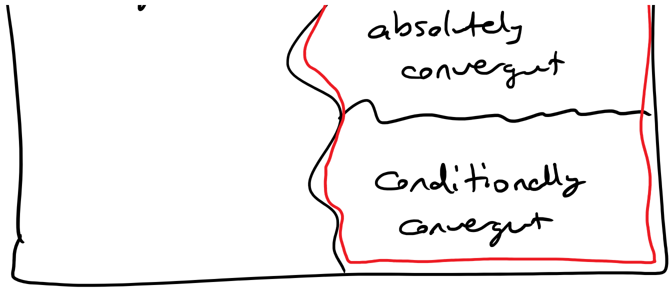
conditionally. (conditionally convergent)

a series $\sum a_n$ is $\to$ divergent

a series $\sum a_n$ is $\to$ convergent $\to$ absolutely

convergent $\to$ conditionally 1





If $\sum_{n=1}^{\infty} a_n$ where $a_n \geq 0$ or $a_n \leq 0$ ultimately, then

convergence $\Longleftrightarrow$ absolute convergence.

Alternating Series Test

$$\sum_{n=1}^{\infty} a_n$$

i) $a_n \cdot a_{n+1} < 0$ for all n.

(Notice: $a_n \neq 0$)

ii) $|a_{n+1}| \leq |a_n|$ for all n.

iii) $\lim_{n \to \infty} a_n = 0$

Then $\sum_{n=1}^{\infty} a_n$ converges.

ex: $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ $\qquad a_n = \frac{(-1)^{n-1}}{n}$

i) $a_n \cdot a_{n+1} = \frac{(-1)^{n-1}}{n} \cdot \frac{(-1)^n}{n+1} = \frac{(-1)^{2n-1}}{n(n+1)}$

$$= \frac{-1}{n(n+1)} < 0 \checkmark$$

ii) $|a_{n+1}| \leq |a_n|$?

$|a_{n+1}| = \left| \frac{(-1)^n}{n+1} \right| = \frac{1}{n+1} \leq |a_n| = \left| \frac{(-1)^{n-1}}{n} \right| = \frac{1}{n}$?

$$|a_{n+1}| = \left| \frac{(-1)^{n}}{n+1} \right| = \frac{1}{n+1} \leq |a_n| = \left| \frac{(-1)^{n-1}}{n} \right| = \frac{1}{n} \; ?$$

Yes. ✓

iii) $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \dfrac{(-1)^{n-1}}{n} = 0$ ✓

(Squeeze Thm: $-\dfrac{1}{n} \leq \dfrac{(-1)^{n-1}}{n} \leq \dfrac{1}{n}$, $\lim \dfrac{1}{n} = \lim \dfrac{-1}{n} = 0$)

$\left(\Rightarrow \lim \dfrac{(-1)^{n-1}}{n} = 0 \right)$

So since (i) – (iii) of the AST are satisfied,

$\sum\limits_{n=1}^{\infty} \dfrac{(-1)^{n-1}}{n}$ converges. (conditionally)

<u>Question:</u> $\sum\limits_{n=1}^{\infty} a_n$. Determine whether it is divergent, is absolutely convergent, or conditionally convergent.

<u>Procedure:</u>
- Use the n^{th} term test for divergence.
- Check for absolute convergence using Integral, Comparison, Limit Comparison, Ratio, Root test.
- Use the Alternating Series Test (as a last resort)

ex: $\sum\limits^{\infty} (-1)^n \dfrac{n^3 - 5}{}$. Diverges / converges / converges ?

ex: $\displaystyle\sum_{n=2}^{\infty} (-1)^n \frac{n^3-5}{n^3+5}$. Diverges / converges absolutely / converges conditionally ?

n^{th} term test: $\displaystyle\lim_{n\to\infty} (-1)^n \frac{n^3-5}{n^3+5}$ does not exist.

So $\displaystyle\sum_{n=2}^{\infty} (-1)^n \cdot \frac{n^3-5}{n^3+5}$ diverges.

ex: $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^3+1}$. Same question as above.

- n^{th} term test for divergence: $\displaystyle\lim_{n\to\infty} \frac{(-1)^{n-1}}{n^3+1} = 0$ (test is inconclusive)

- Check for absolute convergence: $\displaystyle\sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{n^3+1} \right| = \sum_{n=1}^{\infty} \underbrace{\frac{1}{n^3+1}}_{a_n}$

Limit Comparison with $\displaystyle\sum_{n=1}^{\infty} \underbrace{\frac{1}{n^3}}_{b_n}$ (convergent):

$$\lim_{n\to\infty} \frac{a_n\left(\frac{1}{n^3+1}\right.}{b_n\left(\frac{1}{n^3}\right.} = \lim_{n\to\infty} \frac{n^3}{n^3+1} = 1$$

So since $\sum \frac{1}{n^3}$ converges, $\sum \frac{1}{n^3+1}$ converges by LCT.

$\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^3+1}$ converges absolutely.

(AST tells us that $\displaystyle\sum \frac{(-1)^{n-1}}{n^3+1}$ converges.

conditionally? absolutely?)

<u>ex:</u> $\sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{\ln n}$. Same question.

- n^{th} term test : $\lim_{n\to\infty} \frac{(-1)^{n-1}}{\ln n} = 0$ (test is inconclusive)

- Check for absolute convergence

$$\sum_{n=2}^{\infty} \left| \frac{(-1)^{n-1}}{\ln n} \right| = \sum_{n=2}^{\infty} \frac{1}{\ln n}$$

$\frac{1}{\ln n} > \frac{1}{n} > 0$ $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. So

$\sum_{n=2}^{\infty} \frac{1}{\ln n}$ diverges. So $\sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{\ln n}$ <u>does not converge absolutely.</u>

- Use AST: i) $a_n \cdot a_{n+1} = \frac{(-1)^{n-1}}{\ln n} \cdot \frac{(-1)^n}{\ln (n+1)} = \frac{-1}{(\ln n)(\ln (n+1))}$ < 0. ✓

ii) $|a_{n+1}| \leq |a_n|$?

$$\left| \frac{(-1)^n}{\ln (n+1)} \right| = \frac{1}{\ln (n+1)} \leq \left| \frac{(-1)^{n-1}}{\ln n} \right| = \frac{1}{\ln n}$$ ✓

iii) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{(-1)^{n-1}}{\ln n} = 0$ ✓

(i) - (iii) are satisfied, and $\sum^{\infty} \frac{(-1)^{n-1}}{\ln n}$ converges

(i) - (iii) are satisfied, and $\sum_{n=2} \frac{}{\ln n}$ converges

by AST.

$$\sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{\ln n} \text{ converges conditionally.}$$

Error estimate for alternating series

Suppose $\sum a_n$ satisfies (i) - (iii) of the AST.

Then $\sum_{n=1}^{\infty} a_n$ converges by the AST.

$$s = \sum_{n=1}^{\infty} a_n$$

$$s_n = \underbrace{a_1 + a_2 + \cdots + a_n} \quad \text{(partial sum)}$$

$$s \approx s_n$$

$$\left|\overbrace{\underbrace{s}_{\text{actual value}} - \underbrace{s_n}_{\text{approximate value}}}^{\text{error}}\right| \le \left|\overbrace{a_{n+1}}^{\text{first omitted term.}}\right|$$

$s - s_n$ and a_{n+1} have the same sign (+ or −).

ex: How many terms of $\sum_{n=1}^{\infty} \frac{(-1)^n}{1+2^n}$ are needed to compute the sum of the series with an error less than 0.001?

Check: (i) - (iii) of the AST are satisfied. So we can use the error estimate for alternating series.

We want $|\text{error}| < 0.001 = \frac{1}{1000}$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{1+2^n} = \Big[\underset{n=1}{-\frac{1}{3}} + \underset{n=2}{\frac{1}{5}} - \underset{n=3}{\frac{1}{9}} + \underset{n=4}{\frac{1}{17}} - \underset{n=5}{\frac{1}{33}} + \underset{n=6}{\frac{1}{65}} - \underset{n=7}{\frac{1}{129}} + \underset{n=8}{\frac{1}{257}} - \underset{n=9}{\frac{1}{513}}\Big] + \underset{n=10}{\frac{1}{1025}} \cdots$$

s_9 = first nine terms

$$|s - s_9| \leq |a_{10}| = \frac{1}{1025} < \frac{1}{1000}.$$

You need nine terms.