

$$\sum_{n=1}^{\infty} a_n, \quad a_n \geq 0$$

or ultimately $a_n \geq 0$
($n \geq N$)

Integral Test

Let $a_n = f(n)$ f : positive, continuous,
nonincreasing on

Then, $\sum_{n=1}^{\infty} a_n$ and $\int_N^{\infty} f(t) dt$ both

converge or both diverge (to infinity)

Ex: $\sum_{n=1}^{\infty} \frac{1}{n}$: $\frac{1}{n} = f(n)$ where $f(t) = \frac{1}{t}$

is positive, continuous, nonincreasing
on $[1, \infty)$.

$$\begin{aligned} \int_1^{\infty} \frac{1}{t} dt &= \lim_{A \rightarrow \infty} \int_1^A \frac{1}{t} dt = \lim_{A \rightarrow \infty} \ln |t| \Big|_1^A \\ &= \lim_{A \rightarrow \infty} (\ln A - \cancel{\ln 1}) = \infty \end{aligned}$$

So, by Integral Test, $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges

$n=1$ to infinity.

Ex: $\sum_{n=1}^{\infty} \frac{1}{n^2}$

$f(t) = \frac{1}{t^2}$ positive,
continuous,
nonincreasing
on $[1, \infty)$

$$\begin{aligned} \int_1^{\infty} \frac{1}{t^2} dt &= \lim_{A \rightarrow \infty} \int_1^A \frac{1}{t^2} dt = \lim_{A \rightarrow \infty} \left. -\frac{1}{t} \right|_1^A \\ &= \lim_{A \rightarrow \infty} -\frac{1}{A} - \left(-\frac{1}{1}\right) = \lim_{A \rightarrow \infty} \left(1 - \frac{1}{A}\right) \\ &= 1 \end{aligned}$$

By Integral Test, $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, but is $\neq 1$.

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = q \cdot \pi^2 \quad q: \text{rational number}$$

Thm: $\sum_{n=1}^{\infty} \frac{1}{n^p} \begin{cases} \text{converges} & \text{if } p > 1 \\ \text{diverges} & \text{if } p \leq 1 \end{cases}$

(p-test)

$$\left(\int_1^{\infty} \frac{1}{t^p} dt \quad \text{if } p > 0 \right)$$

$$\sum_{n=1}^{\infty} \frac{1}{n^5} \quad \text{converges} \quad (p\text{-test, } p=5).$$

$p > 1$

Comparison Test

$$\underline{0 \leq a_n \leq K \cdot b_n} \quad \text{for some } K > 0$$

(ultimately)

a) If $\sum b_n$ converges, then $\sum a_n$ converges

$$(\sum b_n \text{ converges} \Rightarrow \sum K \cdot b_n \text{ converges})$$

b) If $\sum a_n$ diverges to infinity, then
 $\sum b_n$ also diverges to infinity.

Ex: Test $\sum_{n=1}^{\infty} \frac{1+\cos n}{n^3}$ for convergence.

$$-1 \leq \cos n \leq 1$$

$$0 \leq \underbrace{\frac{1+\cos n}{n^3}}_{a_n} \leq \frac{2}{n^3} = 2 \cdot \underbrace{\frac{1}{n^3}}_{b_n}$$

Since $\sum b_n = \sum \frac{1}{n^3}$ converges (p-test, $p=3$),
 $\sum_{n=1}^{\infty} \frac{1 + \cos n}{n^3}$ converges by Comparison Test.

Ex: Test $\sum_{n=1}^{\infty} \frac{n^2 + 1}{n^5 + 1}$ for convergence.

(Tip: Consider $\sum \frac{n^2}{n^5} = \sum_{n=1}^{\infty} \frac{1}{n^3}$.)
 Converges p-test $p=3$

$$\underline{0} \leq \frac{n^2 + 1}{n^5 + 1} \leq \frac{1}{n^3}$$

$$(n^2 + 1) \cdot n^3 \leq n^5 + 1$$

$$\cancel{n^5} + n^3 \leq \cancel{n^5} + 1 \quad (\text{Does not work.})$$

Try again:

$$\underline{0} \leq \frac{n^2 + 1}{n^5 + 1} \leq \underline{2} \cdot \frac{1}{n^3}$$

$$(n^2 + 1) \cdot n^3 \leq \underline{2} \cdot (n^5 + 1)$$

$$n^5 + n^3 \leq \underline{2} \cdot (n^5 + 1)$$

$$\cancel{n^5} + n^3 \leq \cancel{2} n^5 + 2$$

$$n^3 \leq n^5 + 2. \text{ Holds for } n \geq 1 !$$

$$0 \leq \underbrace{\frac{n^2 + 1}{n^5 + 1}}_{a_n} \leq \underbrace{2}_{K} \cdot \underbrace{\frac{1}{n^3}}_{b_n}$$

Since $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^3}$ converges (p-test, $p=3$),

$\sum_{n=1}^{\infty} \frac{n^2 + 1}{n^5 + 1}$ converges by Comparison Test.

Ex: $\sum_{n=2}^{\infty} \frac{1}{\ln n}$

$$0 < \ln n < n \text{ for } n \geq 2$$

$$\text{So } \frac{1}{\underbrace{\ln n}_{b_n}} > \frac{1}{\underbrace{n}_{a_n}} > 0$$

Since $\sum_{n=2}^{\infty} a_n = \sum_{n=2}^{\infty} \frac{1}{n}$ diverges to infinity,

$\sum_{n=2}^{\infty} b_n = \sum_{n=2}^{\infty} \frac{1}{\ln n}$ diverges to infinity by Comparison Test.

$n = 1, \dots$

Comparison Test.

Limit Comparison Test: $a_n, b_n > 0$,

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = L \quad \text{where } L \in [0, \infty) \text{ or } L = \infty$$

a) If $L < \infty$ and $\sum b_n$ converges, then $\sum a_n$ converges.

b) If $L > 0$ and $\sum b_n$ diverges to infinity, then $\sum a_n$ diverges to infinity.

Ex: Test $\sum_{n=1}^{\infty} \frac{n+5}{n^3-2n+3}$ for convergence.

$$b_n = \frac{n}{n^3} = \frac{1}{n^2}$$

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{\frac{n+5}{n^3-2n+3}}{\frac{1}{n^2}} = \lim_{n \rightarrow \infty} \frac{n^3+5n^2}{n^3-2n+3} = 1 > 0$$

By Limit Comparison, since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges,

$\sum_{n=1}^{\infty} \frac{n+5}{n^3-2n+3}$ also converges.

$$n=1$$

$$L = \lim_{n \rightarrow \infty} \frac{a_n}{b_n} \quad L \in (0, \infty)$$

Then $\sum a_n$ and $\sum b_n$ both converge or both diverge.

Ex: $\sum_{n=1}^{\infty} \underbrace{\frac{1}{1+\sqrt{n}}}_{a_n}$. Take $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{\frac{1}{1+\sqrt{n}}}{\frac{1}{\sqrt{n}}} = \lim_{n \rightarrow \infty} \frac{\sqrt{n}}{1+\sqrt{n}} = 1 > 0$$

Since $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges (p -test, $p = \frac{1}{2}$),

$\sum_{n=1}^{\infty} \frac{1}{1+\sqrt{n}}$ diverges by Limit Comparison.

Ratio Test : $a_n > 0$ (ultimately)

$$\rho = \lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} \text{ exists or is } \infty.$$

a) If $0 \leq \rho < 1$, then $\sum_{n=1}^{\infty} a_n$ converges

b) If $1 < p \leq \infty$, then $\lim_{n \rightarrow \infty} a_n = \infty$

and $\sum_{n=1}^{\infty} a_n$ diverges to infinity.

c) If $p=1$, the test is inconclusive.

Ex: $\sum_{n=1}^{\infty} \frac{1}{n}$ $\rho = \lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \lim_{n \rightarrow \infty} \frac{\frac{1}{n+1}}{\frac{1}{n}} = \lim_{n \rightarrow \infty} \frac{n}{n+1} = 1$

diverges

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \quad \rho = \lim_{n \rightarrow \infty} \left(\frac{n}{n+1} \right)^2 = 1$$

Converges

Ex: $\sum_{n=1}^{\infty} \frac{4^n}{n!}$. Test for convergence using Ratio Test.

$$\rho = \lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \lim_{n \rightarrow \infty} \frac{\frac{4^{n+1}}{(n+1)!}}{\frac{4^n}{n!}} = \lim_{n \rightarrow \infty} \frac{4}{n+1} = 0$$

$$\sum_{n=1}^{\infty} \frac{4^n}{n!} \text{ converges by Ratio Test.}$$

Ex: $\sum_{n=1}^{\infty} \frac{n^3}{5^n}$. $\rho = \lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \lim_{n \rightarrow \infty} \frac{\frac{(n+1)^3}{5^{n+1}}}{\frac{n^3}{5^n}}$

$$= \lim_{n \rightarrow \infty} \frac{(n+1)^3}{n^3} \cdot \frac{1}{5} = \frac{1}{5} < 1$$

$\sum_{n=1}^{\infty} \frac{n^3}{5^n}$ converges by Ratio Test

Ex: $\sum_{n=1}^{\infty} \frac{(3n)!}{(n!)^3}$

$$\rho = \lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \lim_{n \rightarrow \infty} \frac{\frac{(3n+3)!}{[(n+1)!]^3}}{\frac{(3n)!}{(n!)^3}} = \lim_{n \rightarrow \infty} \frac{(3n+3) \cdot (3n+2) \cdot (3n+1)}{(n+1)^3} = \lim_{n \rightarrow \infty} \frac{27n^3 + \dots}{n^3 + \dots} = 27 > 1$$

$\sum_{n=1}^{\infty} \frac{(3n)!}{(n!)^3}$ diverges by Ratio Test.

Root Test: $a_n > 0$ (ultimately)

Suppose $\sigma = \lim_{n \rightarrow \infty} (a_n)^{1/n} = \lim_{n \rightarrow \infty} \sqrt[n]{a_n}$ exists or is ∞

(Then: $\sigma \in [0, \infty)$ or $\sigma = \infty$)

a) If $0 \leq \sigma < 1$, $\sum a_n$ converges.

b) If $1 < \sigma \leq \infty$, $\sum a_n$ diverges to infinity

($\lim_{n \rightarrow \infty} a_n = \infty$)

c) If $\sigma = 1$, the test is inconclusive.

∞ n

Ex: Test $\sum_{n=1}^{\infty} \frac{2^n}{n^n}$ for convergence using Root Test.

$$\rho = \lim_{n \rightarrow \infty} \sqrt[n]{a_n} = \lim_{n \rightarrow \infty} \sqrt[n]{\frac{2^n}{n^n}} = \lim_{n \rightarrow \infty} \frac{2}{n} = 0 < 1$$

By Root Test, $\sum_{n=1}^{\infty} \frac{2^n}{n^n}$ is convergent.