

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \dots$$

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$$

$$\sum_{n=3}^{\infty} \frac{1}{n^2} = \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \dots$$

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{4^{n+1}} = \frac{1}{64} - \frac{1}{256} + \frac{1}{1024} + \dots$$

$$\sum_{n=2}^{\infty} \frac{1}{n} = \sum_{n=-3}^{\infty} \frac{1}{n+5}$$

decrease of 5

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \sum_{n=4}^{\infty} \frac{1}{(n-3)^2}$$

increased by 3

$$= 1 + \frac{1}{4} + \frac{1}{9} + \dots = 1 + \frac{1}{4} + \frac{1}{9} + \dots$$

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$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \dots$$

$$S_n = a_1 + a_2 + \dots + a_n$$

S_n : n^{th} partial sums S_n is a real number

$\{S_n\}_{n=1}^{\infty}$: sequence of partial sums

Defn: $\sum_{n=1}^{\infty} a_n$ converges to s if
(s : real number)

$$\lim_{n \rightarrow \infty} S_n = s.$$

We denote $\sum_{n=1}^{\infty} a_n = s$

$\left(\sum_{n=1}^{\infty} a_n \text{ diverges if } \{S_n\} \text{ diverges.} \right)$

 $\begin{matrix} \text{to } \infty \\ \text{or} \\ \text{to } -\infty \end{matrix}$

 $\begin{matrix} \text{to } \infty \\ \text{or} \\ \text{to } -\infty \end{matrix}$

Harmonic Series:

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$$

Claim: The harmonic series diverges to infinity

$$1 + \underbrace{\frac{1}{2}}_{\geq \frac{1}{2}} + \underbrace{\frac{1}{3} + \frac{1}{4}}_{\frac{7}{12} \geq \frac{1}{2}} + \underbrace{\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}}_{\geq \frac{1}{2}} + \underbrace{\frac{1}{9} + \dots + \frac{1}{16}}_{\geq \frac{1}{2}} + \dots$$

We can add as many copies of $\frac{1}{2}$ to the partial sums.

$$\lim_{n \rightarrow \infty} S_n = \infty.$$

$$S_{\infty} = \sum_{n=1}^{\infty} \frac{1}{n} = \infty. \quad (\text{diverges to infinity})$$

Telescoping Series:

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \dots$$

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$$

$$S_N = \sum_{n=1}^N \frac{1}{n(n+1)} = \left(\frac{1}{1} - \frac{1}{2} \right) + \left(\frac{1}{2} - \frac{1}{3} \right) + \dots + \left(\frac{1}{N-1} - \frac{1}{N} \right) + \left(\frac{1}{N} - \frac{1}{N+1} \right)$$

$$S_N = 1 - \frac{1}{N+1} \quad \text{Cancels out}$$

$$\lim_{N \rightarrow \infty} S_N = \lim_{N \rightarrow \infty} \left(1 - \frac{1}{N+1} \right) = 1$$

$$\text{So } \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1 \quad \square$$

Geometric Series

$$\sum_{n=1}^{\infty} a \cdot \underbrace{r^{n-1}}_{\substack{\text{common} \\ \text{ratio}}} = \underbrace{a}_{\text{initial term}} + ar + ar^2 + ar^3 + \dots$$

$$r = \text{common ratio} = \frac{a_{n+1}}{a_n}$$

$$S_n = a + \cancel{ar} + \cancel{ar^2} + \dots + \cancel{ar^{n-1}}$$

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$$\underline{\underline{r S_n = \cancel{ar} + \cancel{ar^2} + \dots + \cancel{ar^{n-1}} + a \cdot r^n}}$$

$$S_n - r \cdot S_n = (1-r) \cdot S_n = a - a \cdot r^n$$

$$\text{If } r \neq 1, \quad S_n = \frac{a(1-r^n)}{1-r}$$

Thm:

$$\sum_{n=1}^{\infty} ar^{n-1} \begin{cases} \rightarrow \text{converges to } 0 \text{ if } a=0 \\ \rightarrow \text{Converges to } \frac{a}{1-r} \text{ if } |r| < 1 \\ \rightarrow \text{diverges to } \infty \text{ if } r \geq 1, a > 0 \\ \rightarrow \text{diverges to } -\infty \text{ if } r \geq 1, a < 0 \\ \rightarrow \text{diverges if } r \leq -1, a \neq 0 \end{cases}$$

$$\underline{\text{Ex:}} \quad \sum_{n=1}^{\infty} (-2)^n = \underbrace{-2 + 4}_{S_2} - 8 + 16 - 32 \dots$$

$$S_1 = -2 \quad S_2 = 2 \quad S_3 = -6$$

$$S_3 = -6 \quad S_4 = 10 \quad S_5 = -22 \quad \text{etc.}$$

$\{S_n\}$ diverges.

$$\underline{\text{Ex:}} \quad \sum_{n=1}^{\infty} \frac{1}{3^{n+1}} = \underbrace{\frac{1}{9}}_a + \frac{1}{27} + \frac{1}{81} + \dots$$

$$a = \frac{1}{9} \quad r = \frac{1}{3} \quad \text{Since } |r| < 1,$$

the series converges to

$$\frac{a}{1-r} = \frac{\frac{1}{9}}{1 - \frac{1}{3}} = \frac{\frac{1}{9}}{\frac{2}{3}} = \frac{1}{6}$$

Ex: $\sum_{n=1}^{\infty} (-1)^n = \underbrace{-1 + 1}_{S_2} - 1 + 1 \dots$

$\underbrace{\hspace{10em}}_{S_3}$

$$S_1 = -1 \quad S_2 = 0$$

$$S_3 = -1 \quad S_4 = 0 \quad \text{etc.}$$

$\{S_n\}$ diverges.

Ex: $0.\overline{41} = 0.414141\dots$

$$= 0.41 + 0.0041 + 0.000041 + \dots$$

$$= \frac{41}{100} + \frac{41}{100^2} + \frac{41}{100^3} + \dots$$

$$a = \frac{41}{100} \quad r = \frac{1}{100} \quad |r| < 1$$

Converges to $\frac{a}{1-r} = \frac{\frac{41}{100}}{\frac{99}{100}} = \frac{41}{99}$

$$0.\overline{41} = \frac{1}{1-r} - \frac{\frac{99}{100}}{99}$$

$$0.\overline{41} = \frac{41}{99}$$

Thm: If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim a_n = 0$.
^{nth term}
^{test for divergence}

Alternatively, if $\lim a_n \neq 0$ or it doesn't exist, then $\sum_{n=1}^{\infty} a_n$ diverges.

(This theorem is one-directional.)

Ex: $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges but $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$!

Ex: $\sum_{n=1}^{\infty} \frac{n^2+1}{2n^2+1}$ diverges

Since $\lim_{n \rightarrow \infty} \frac{n^2+1}{2n^2+1} = \frac{1}{2} \neq 0$.

Ex: $\sum_{n=1}^{\infty} \sin(n)$ diverges since

$\lim \sin(n)$ does not exist.

$$n \rightarrow \infty$$

Thm: $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{n=N}^{\infty} a_n$ converges for any integer $N \geq 1$.

Thm: If $\{a_n\}$ is ultimately positive, then

$\sum_{n=1}^{\infty} a_n$ $\rightarrow$ partial sums are bounded above and $\sum a_n$ converges

$\rightarrow$ partial sums are not bounded above and $\sum a_n$ diverges to infinity.

Thm: $\sum_{n=1}^{\infty} a_n = A$ $\sum_{n=1}^{\infty} b_n = B$ c : any constant

a) $\sum_{n=1}^{\infty} c \cdot a_n = c \cdot A$

b) $\sum_{n=1}^{\infty} (a_n \mp b_n) = A \mp B$

c) If $a_n \leq b_n$ for all n then

$$A \leq B.$$

$$\underline{\text{Ex:}} \quad \sum_{n=1}^{\infty} \frac{3+2^n}{3^{n+1}} = \sum_{n=1}^{\infty} \left(\frac{3}{3^{n+1}} + \frac{2^n}{3^{n+1}} \right)$$

$$\sum_{n=1}^{\infty} \frac{3}{3^{n+1}} = \sum_{n=1}^{\infty} \frac{1}{3^n} = \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \dots = \frac{a}{1-r}$$

$$\left(a = \frac{1}{3} \quad r = \frac{1}{3} \right) = \frac{\frac{1}{3}}{1 - \frac{1}{3}} = \frac{1/3}{2/3} = \frac{1}{2}$$

$$\sum_{n=1}^{\infty} \frac{2^n}{3^{n+1}} = \frac{2}{9} + \frac{4}{27} + \frac{8}{81} + \dots = \frac{a}{1-r}$$

$$\left(a = \frac{2}{9} \quad r = \frac{2}{3} \right) = \frac{\frac{2}{9}}{1 - \frac{2}{3}} = \frac{2/9}{1/3} = \frac{2}{3}$$

$$\hookrightarrow \sum_{n=1}^{\infty} \frac{3+2^n}{3^{n+1}} = \frac{1}{2} + \frac{2}{3} = \frac{7}{6} \quad \square$$