$$\{1, 2, 3, 4, 5, 6, 7, \ldots\}$$

$$\{2, 2, 2, \ldots\}$$

$$\left\{-\frac{1}{4}, \frac{1}{16}, -\frac{1}{64}, \frac{1}{256}, \ldots\right\}$$

$$\{a_1, a_2, a_3, a_4, \ldots\} \qquad a_n: \text{general term}$$

$$f: \mathbb{N} \longrightarrow \mathbb{R} \qquad \mathbb{N} = \{1, 2, 3, \ldots\}$$

$$f(n) = a_n \qquad n: \text{index}$$

$$\{a_n\}_{n=1}^{\infty} \qquad a_n = \frac{1}{2^n} \qquad a_1 = \frac{1}{2^1} = \frac{1}{2} \qquad a_2 = \frac{1}{2^2} = \frac{1}{4}$$

$$a_3 = \frac{1}{2^3} = \frac{1}{8} \quad \text{etc.}$$

$$\{(-1)^n\}_{n=1}^{\infty} = \{(-1)^1 = -1, (-1)^2 = 1, -1, 1, -1, 1, \ldots\}$$

$$\{(-1)^{n+1}\}_{n=1}^{\infty} = \{(-1)^{1+1} = (-1)^2 = 1, -1, 1, -1, \ldots\}$$

$$\{\cos(n\pi)\}_{n=1}^{\infty} = \{\cos\pi, \cos 2\pi, \cos 3\pi, \cos 4\pi, \ldots\}$$

$$= \{-1, 1, -1, 1, -1, \ldots\}$$

$(-1)^n = \cos(n\pi)$

$a_1 = 1 \qquad a_2 = 1 \qquad a_{n+2} = a_{n+1} + a_n$

$a_3 = 1+1 = 2 \qquad a_4 = 2+1 = 3 \qquad a_5 = 3+2 = 5$

$a_6 = 3+5 = 8 \qquad a_7 = 5+8 = 13 \qquad a_8 = 8+13 = 21$

etc.

Fibonacci sequence : $\{1, 1, 2, 3, 5, 8, 13, 21, \ldots\}$

(recursively defined sequence)

Terms to describe sequences

- $\{a_n\}$ is bounded below if

  $a_n \geq L$ for some real number L

  L: lower bound for $\{a_n\}$

- $\{a_n\}$ is bounded above if

  $M \geq a_n$ for some real number M

  M: upper bound for $\{a_n\}$

- $\{a_n\}$ is bounded if it is bounded above and bounded below.

- $\{a_n\}$ is bounded if there exists $K > 0$ such that $|a_n| \leq K$.

- $\{a_n\}$ is positive if $a_n \geq 0$

- $\{a_n\}$ is negative if $a_n \leq 0$

Ex: $a_n = 0$ $\quad \{a_n\}$ is both positive and negative.

Ex: $a_n = (-1)^n$ $\quad \{a_n\}$ is neither positive nor negative

$$(-1)^n \not\geq 0 \qquad (-1)^n \not\leq 0$$

- $\{a_n\}$ is increasing if $a_{n+1} \geq a_n$
- $\{a_n\}$ is decreasing if $a_{n+1} \leq a_n$.

Ex: $a_n = 7$ $\quad \{a_n\}$ is both increasing and decreasing

Ex: $a_n = (-1)^n$ $\quad \{a_n\}$ is neither increasing nor decreasing

- $\{a_n\}$ is monotonic if it is either increasing or decreasing

- $\{a_n\}$ is alternating if $a_{n+1} \cdot a_n < 0$

Ex: $\{(-1)^n\}$ is alternating

- P: one of these properties

ultimately P : P is true after a certain point in the sequence

Ex: $\{1, \underbrace{2, 3}_{2\cdot 3 = 6 > 0}, 4, \underbrace{-1, 1, -1, 1, -1, 1, \ldots}_{\text{alternating}}\}$

is not alternating, but ultimately alternating

Ex: $\left\{ \frac{n+1}{n+4} \right\}_{n=1}^{\infty} = \left\{ \frac{2}{5}, \frac{3}{6}, \frac{4}{7}, \frac{5}{8}, \frac{6}{9}, \frac{7}{10}, \ldots \right\}$

increasing, not decreasing, monotonic

bounded below, bounded above $\left(\frac{n+1}{n+4} \leq 1 \right)$

bounded

positive $\left(\frac{n+1}{n+4} \geq 0 \right)$, not negative

not alternating

$a_n = \frac{n+1}{n+4}$ $\qquad a_{n+1} \geq a_n$? ← True

$$\frac{n+2}{n+5} \geq \frac{n+1}{n+4} \ ?$$

$$(n+2)\cdot(n+4) \geq (n+1)\cdot(n+5) \ ?$$

$$\cancel{n^2} + \cancel{6n} + 8 \geq \cancel{n^2} + \cancel{6n} + 5 \ ?$$

$$8 \geq 5 \ ? \quad \text{True.}$$

$f(x) = \frac{x+1}{x+4}$ $\qquad f'(x) = \frac{1\cdot(x+4) - 1\cdot(x+1)}{(x+4)^2}$

$$= \frac{3}{(x+4)^2} > 0 \quad \text{if } x > 1.$$

So $f(x)$ is an increasing function.

So $\{a_n\}$ is an increasing sequence

Ex: $\left\{ \dfrac{n^2}{2^n} \right\}_{n=1}^{\infty} = \left\{ \underbrace{\dfrac{1}{2}, 1, \dfrac{9}{8}}_{\text{inc.}}, 1, \dfrac{25}{32}, \dfrac{36}{64}, \dfrac{49}{128}, \dots \right\}$

dec.

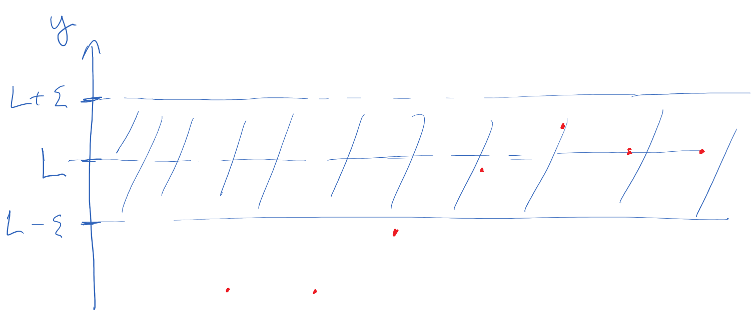
- positive, not negative
- not increasing, not decreasing, not monotonic
- ultimately decreasing (ultimately monotonic)
- bounded below $(a_n \geq 0)$
- bounded above $\left(a_n \leq \dfrac{9}{8}\right)$
- bounded
- not alternating

Convergence of Sequences

Definition: $\{a_n\}$ converges to L, $\lim\limits_{n \to \infty} a_n = L$,

if: for every $\varepsilon > 0$ there exists a N such that whenever $n \geq N$, $|a_n - L| < \varepsilon$

"We can make the terms a_n as close to L as we wish by considering the sequence from N on."



$$|a_n - L| < \varepsilon \iff a_n \in (L-\varepsilon, L+\varepsilon)$$

Ex: Show that $\lim_{n\to\infty} \frac{1}{n} = 0$.

Pick $\varepsilon > 0$. We want: a positive integer N such that if $n \geq N$, then $|a_n - L| < \varepsilon$.

i.e. $\left|\frac{1}{n} - 0\right| < \varepsilon$.

$N = \left\lceil \frac{1}{\varepsilon} \right\rceil + 1$ (smallest integer above $\frac{1}{\varepsilon}$). $N > \frac{1}{\varepsilon}$

In that case, if $n \geq N$, then $\left|\frac{1}{n} - 0\right| < \varepsilon$:

if $n \geq N$, then $\frac{1}{n} < \frac{1}{N}$

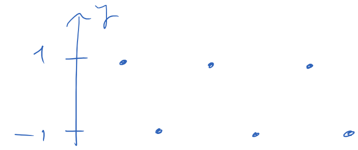
$$\left|\frac{1}{n} - 0\right| \leq \frac{1}{N} < \varepsilon$$

$$\Longrightarrow \left|\frac{1}{n} - 0\right| < \varepsilon \qquad \square$$

$$\boxed{\lim_{n\to\infty} \frac{1}{n} = 0} \qquad \boxed{\lim \frac{1}{n} = 0}$$

Ex: $\left\{\frac{n+1}{n+4}\right\}$ converges to 1.

Ex: $\{(-1)^n\}$ diverges.



Ex: $\{n\}$ diverges (to infinity)

Ex: $\{-n^2\}$ diverges (to negative

infinity)

$$\lim n = \infty \qquad \lim -n^2 = -\infty$$

Thm: If $\lim_{x\to\infty} f(x) = L$ then $\lim f(n) = L$

Ex: $\lim \frac{1}{n^2} = 0$ because $\lim_{x\to\infty} \frac{1}{x^2} = 0$

$(-1)^n \rightsquigarrow (-1)^x$ is not defined.

Rules for Limits: $\lim a_n = L$

$\lim b_n = M$

$\lim (a_n + b_n) = L + M$ $\qquad$ c: constant

$\lim (a_n - b_n) = L - M$

$\lim (a_n \cdot b_n) = L \cdot M$

$\lim \left(\frac{a_n}{b_n}\right) = \frac{L}{M}$ provided that $M \neq 0$ and $b_n \neq 0$.

$\lim (c \cdot a_n) = c \cdot L$

If $a_n \leq b_n$ ultimately, then $L \leq M$

(Squeeze Theorem) If $a_n \leq b_n \leq c_n$, and $\lim a_n = \lim c_n = L$

then $\lim b_n = L$

Ex: Find $\lim \frac{\sin(n)}{n^2+1}$

$$-1 \leq \sin(n) \leq 1$$

$$\underbrace{-\frac{1}{n^2+1}} \leq \frac{\sin(n)}{n^2+1} \leq \underbrace{\frac{1}{n^2+1}}$$

$\lim \frac{-1}{n^2+1} = 0$ $\lim \frac{1}{n^2+1} = 0$

By Squeeze Theorem, $\lim \frac{\sin(n)}{n^2+1} = 0$

(Note: $\lim \frac{1}{n^2+1} = 0$ because $\lim_{x\to\infty} \frac{1}{x^2+1} = 0$)

$\left(\lim \frac{-1}{n^2+1} = -\left(\lim \frac{1}{n^2+1}\right) = -0 = 0 \right)$

Ex: $\lim\left(\sqrt{n^2+4n} - n\right) = \lim \underbrace{\frac{\sqrt{n^2+4n}+n}{\sqrt{n^2+4n}+n}}_{=1} \cdot \left(\sqrt{n^2+4n} - n\right)$

$$= \lim \frac{(\cancel{n^2}+4n)-(\cancel{n})^2}{\sqrt{n^2+4n}+n} = \lim \frac{4n}{\sqrt{n^2+4n}+n} = \lim \frac{4\cancel{n}}{\cancel{n}\cdot\sqrt{1+\frac{4}{n}}+\cancel{n}}$$

$$= \lim \frac{4}{\sqrt{1+\frac{4}{n}}+1} = \frac{4}{\sqrt{1+0}+1} = \frac{4}{1+1} = 2$$

$\lim \frac{4}{n} = 0$

$\lim \sqrt{1+\frac{4}{n}} = \sqrt{1+\lim\left(\frac{4}{n}\right)} = \sqrt{1+0} = 1$

because $\sqrt{\ }$ is a continuous function.

$\lim \sin\left(\frac{1}{n}\right) = \sin\left(\lim \frac{1}{n}\right) = \sin 0 = 0$

Theorem: If $\{a_n\}$ converges, it is bounded.

Note: Bounded sequences are not necessarily convergent.

(example: $\{(-1)^n\}$ is bounded, not convergent)

Completeness property of real numbers:

$$\text{Bounded} + \underbrace{\text{monotonic}}_{\text{essential}} \Longrightarrow \text{convergent}$$

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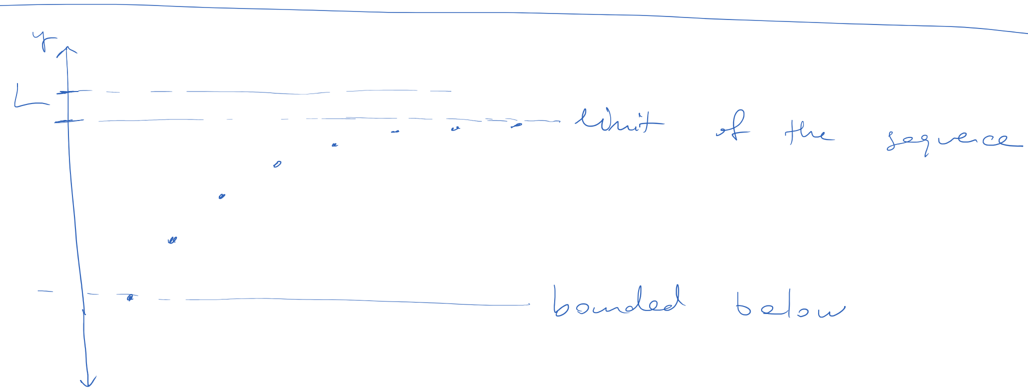
ex: bounded, increasing



"Bounded, monotonic sequences converge."

Ex: Let $a_1 = 1$ $\quad a_{n+1} = \sqrt{1+2a_n}$ $\quad (n = 1, 2, 3, \ldots)$

Show that this sequence converges; find its limit.

Claim 1: $\{a_n\}$ is bounded.

Bounded below: $a_n \geq 0$ $\quad$ Induction: $a_1 \geq 0$

$a_n \geq 0 \Rightarrow 1 + 2a_n \geq 0$

$\Rightarrow \sqrt{1+2a_n} = a_{n+1} \geq 0.$

$a_1 = 1$

$a_2 = \sqrt{3}$

$a_3 = \sqrt{1+2\sqrt{3}}$ etc.

Bounded above: Pick a large number: 10.

$a_n \leq 10$: Induction: $a_1 \leq 10.$

$a_2 = \sqrt{3}$

$a_3 = \sqrt{1+2\sqrt{3}}$ etc.

$a_n \le 10$: Induction: $a_1 \le 10$.

$a_n \le 10 \Rightarrow 1+2a_n \le 21 \le 100$

$\Rightarrow \sqrt{1+2a_n} \le 10$

$\Rightarrow a_{n+1} \le 10$.

$\Rightarrow \{a_n\}$ is bounded.

Increasing : Induction: $a_2 \ge a_1$ True.

$(\sqrt{3} \ge 1)$

increasing, hence monotonic

$a_{n+1} \ge a_n \Rightarrow 1+2a_{n+1} \ge 1+2a_n$

$\Rightarrow \sqrt{1+2a_{n+1}} \ge \sqrt{1+2a_n}$

$\Rightarrow a_{n+2} \ge a_{n+1}$.

So $\{a_n\}$ is bounded and monotonic. Hence it converges

$L = \lim a_n$.

$a_{n+1} = \sqrt{1+2a_n}$ Take limits of both sides as $n \to \infty$:

$L = \sqrt{1+2L}$ $\quad \left(\lim\limits_{n\to\infty} a_{n+1} = L\right)$

$L^2 = 1+2L$

$L^2 - 2L - 1 = 0$

$L = \dfrac{2 \mp \sqrt{4+4}}{2} = \dfrac{2 \mp \sqrt{8}}{2} = 1 \mp \sqrt{2}$

$\boxed{L = 1+\sqrt{2}}$ $\quad \cancel{L = 1-\sqrt{2}}$

impossible $L<0$.

$\lim a_n = 1+\sqrt{2}$ $\quad a_n \ge 0$ Then $L \ge 0$.

Thm: $\{a_n\}$ (ultimately) increasing decreasing, then it is either bounded above below, hence convergent, or not bounded above below, hence divergent to (negative) infinity.

(Similar statement for (ultimately) decreasing sequences)

Thm: a) If x is a real number with $|x| < 1$, then $\lim x^n = 0$.

$$\left(\underline{\text{Ex}}: \lim \frac{1}{2^n} = 0 \right)$$

b) If x is a real number then

$$\lim \frac{x^n}{n!} = 0$$

"Factorials grow faster than exponentials."

Ex: $$\lim \frac{2^n + 3^n}{7^n} = \lim \left(\frac{2}{7}\right)^n + \left(\frac{3}{7}\right)^n$$

$$\lim \left(\frac{2}{7}\right)^n = \lim \left(\frac{3}{7}\right)^n = 0 \quad \text{So} \quad \lim \left(\frac{2}{7}\right)^n + \left(\frac{3}{7}\right)^n = 0.$$